I 1

$\boxed{\text{Ising Model}}$

One of most simple examples of interacting

statistical mechanics system

$$E = -J \sum_{\langle ij \rangle} S_i S_j$$

"spin" degree of freedom

$S_j = \pm 1$ living on site j

near neighbors sites on some lattice

Illustrates many ideas of interacting stat mech in simple context (both methods and concepts)

Example $N=4$ site linear chain pbc

$$E = -J(S_1 S_2 + S_2 S_3 + S_3 S_4 + S_4 S_1)$$

S_1 S_2 S_3 S_4	E	
+ + + +	$-4J$	"Ferromagnetic" configurations have lowest E
− − − −	$-4J$	
+ + + −	ϕ	
+ + − +	ϕ	
+ − + +	ϕ	
− + + +	ϕ	
+ + − −	ϕ	
+ − + −	$+4J$	"antiferromagnetic have highest energy"
− + − +	$+4J$	

I2

$$Z = \sum_{S_1 S_2 S_3 S_4} e^{-\beta E} = 2e^{4\beta J} + 12 + 2e^{-4\beta J}$$

Competition between $\langle E \rangle$ minimization and entropy

to minimize $F = \langle E \rangle - TS$

controls balance of priority between $\langle E \rangle$ and S

picture of phase transition: T_c is "critical temperature" dividing whether $\langle E \rangle$ wins or S wins.

Phase transition: $\boxed{\text{sharp}}$ change in behavior

Z or a derivative thereof is nonanalytic.

How could $\sum e^{-\beta E}$ be nonanalytic ?!?!

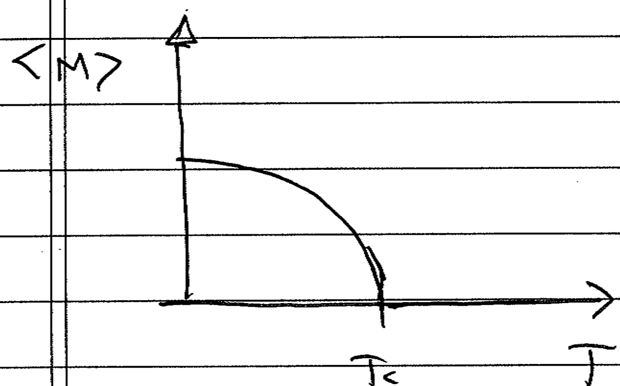
sum must be infinite

I3

Simple example

$1+x+x^2+x^3+\dots x^{20}$ is analytic

but infinite sum is $1/1-x$ which has pole at $x=1$

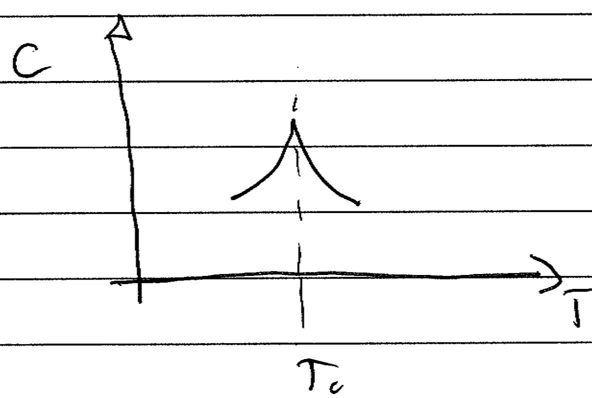


$\langle M\rangle \equiv Z^{-1}\sum m e^{-\beta E}$

$\frac{1}{N}\sum_i S_i$

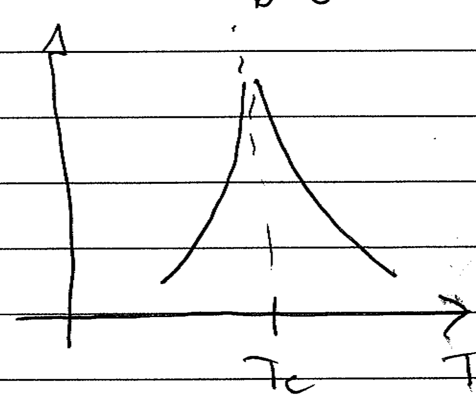
$M\sim (T_c-T)^\beta$ ← critical exponent <u>not</u> $1/k_BT$

for T near T_c



$C\sim (T_c-T)^\alpha$

$\chi = \frac{d\langle M\rangle}{dB}\Big|_{B=0} \sim (T-T_c)^\gamma$



I4

Magnetic field

$$E = -J \sum_{\langle ij \rangle} S_i S_j - B \sum S_i$$

$$\langle M \rangle = Z^{-1} \sum e^{-\beta E} M$$

M

$$= Z^{-1} \sum e^{\beta J \sum S_i S_j + \beta B \sum S_i} M$$

$$\frac{d \langle M \rangle}{dB} = Z^{-1} \sum e^{\beta J \sum S_i S_j + \beta B \sum S_i} \beta M^2 -$$

term from differentiating B in Z^{-1}

$$d\langle M \rangle / dB = \beta \left[\langle M^2 \rangle - \langle M \rangle^2 \right]$$

N.B. It is very weird that $\langle M \rangle \neq 0$ because Ising model has global symmetry $S_i \to -S_i \quad \forall i$

$E \to E$

so configurations come in pairs $\pm M$ same E.

Such Spontaneous symmetry breaking is very mysterious!

I5 A

Connections to HEP

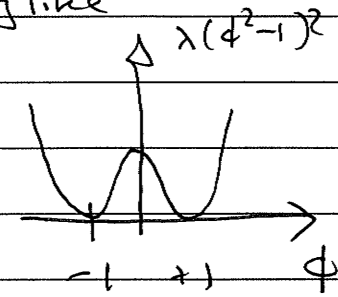
(1) ϕ^4 field theory

$-\infty < \phi_i < \infty$ continuous variable

$$E = -J \sum \phi_i \phi_j + \lambda \sum (\phi_i^2 - 1)^2$$

gives Ising like character

Key issue: Does ϕ get "vacuum expectation value" $\langle \phi \rangle \neq 0$?



Like magnetic phase transition $\langle M \rangle \neq 0$ at low T.

Note also $d\phi/_{dx} = 1/_{dx}[\phi(x+dx) - \phi(x)]$

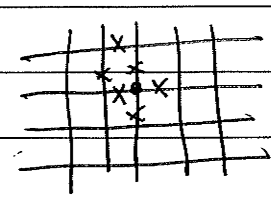
so $(d\phi/_{dx})^2 \sim \phi(x+dx)\phi(x) \sim \phi_i \phi_j$

So really QFT looks at $S = \frac{1}{2}\int (d\phi/_{dx})^2 dx + \lambda \int (\phi^2-1)^2 dx$

(2) Ising model has global symmetry $S_i \to -S_i$ $\boxed{\forall i}$

"Wegner" Ising model $E = -J \sum_{\square} S_i S_j S_k S_e$ local

$S_i = \pm 1$ still but S_i live on links of square lattice



pick any vertex • and flip Ising variables on the emanating links. E is unchanged.

15B

Note Ising is simpler than Heisenberg model we encountered in 215B

$$E \to \hat{H} = +J/\hbar^2 \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

more commonly in solids this sign favors AF

$$= +J/\hbar^2 \sum_{\langle ij \rangle} S_i^z S_j^z + \tfrac{1}{2}\left(S_i^+ S_j^- + S_i^- S_j^+\right)$$

(spin ½) operators

$$\hat{H}\,|++++\rangle = J\,|++++\rangle$$

$$\hat{H}\,|+++-\rangle = \phi\,|+++-\rangle + \frac{J}{2}\,|++-+\rangle + \frac{J}{2}\,|-+++\rangle$$

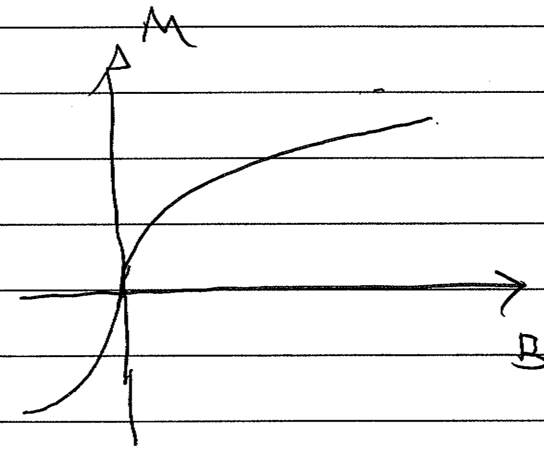
Ising is just this term

Mix states
$\Rightarrow$ must diagonalize matrix to get energy eigenvalues

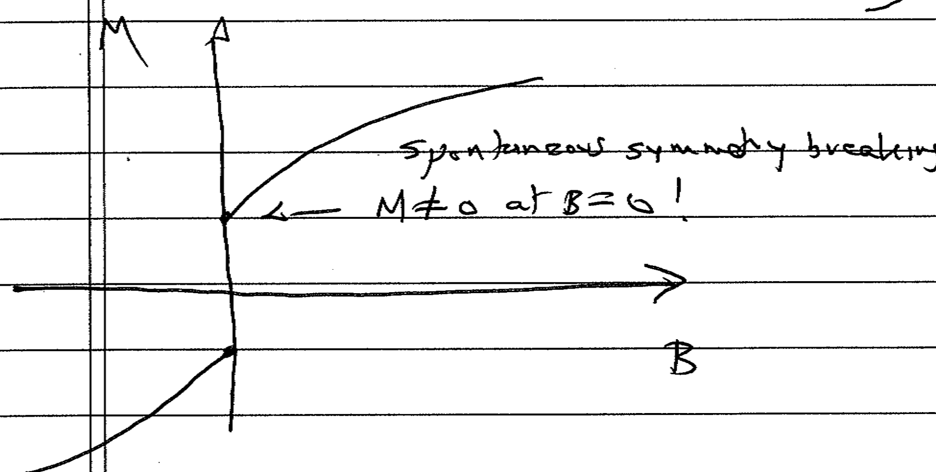
Another critical exponent is δ

$T > T_c$

Right on dividing line



$T = T_c$



$T < T_c$

$M \sim B^{1/\delta}$

$\chi = \dfrac{dM}{dB}$ diverges

(as noted earlier)

It is sort of plausible that χ should diverge at T_c: T_c is the place where system decides spontaneously to magnetize, one might expect its response to an external field to be very strong if already it wants to order on its own!

#7

We can solve Ising model in $d=1$ by "transfer matrix" method

$$Z = \sum_{S_1 S_2 .. S_N} e^{\beta J \sum_{\langle ij \rangle} S_i S_j}$$

$$= \sum_{S_1 S_2 .. S_N} e^{\beta J S_1 S_2} e^{\beta J S_2 S_3} \cdots e^{\beta J S_N S_1}$$

$$\text{Define } M(S, S') = e^{\beta J S S'} \Rightarrow \begin{array}{c c} & \begin{array}{cc} S'=1 & S'=-1 \end{array} \\ \begin{array}{c} S=1 \\ S=-1 \end{array} & \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix} \end{array}$$

$$Z = \sum_{S_1 S_2 .. S_N} \underbrace{M(S_1 S_2) M(S_2 S_3)}\ \cdots M(S_N S_1)$$

$\sum_{S_2}$ of this gives $M^2(S_1 S_3)$

Then $\sum_{S_3} M^2(S_1 S_3) M(S_3 S_4) = M^3(S_1 S_4)$

$$Z = \sum_{S_1} M^N(S_1 S_1) = \mathrm{Tr}\, M^N$$

$\mathrm{Tr}\, M^N = \lambda_1^N + \lambda_2^N$ $\quad \lambda_1 \lambda_2$ eigenvalues of M

$= 2\cosh\beta J,\ 2\sinh\beta J$

I 8

$$Z = (2\cosh\beta J)^N + (2\sinh\beta J)^N$$

Well behaved as $N \to \infty$.

No phase transition in $d=1$ (But there is one in $d \geq 2$)

check $N=4$

$$(e^{\beta J} + e^{-\beta J})^4 + (e^{\beta J} - e^{-\beta J})^4$$

$$= e^{4\beta J} + 4e^{2\beta J} + 6e^{0} + 4e^{-2\beta J} + e^{-4\beta J}$$

$$+ \qquad - \qquad\qquad -$$

$$2e^{4\beta J} + 12e^{0} + 2e^{-4\beta J}$$

I 8A

More on transfer matrix in $d=1$

① $Z = (2\cosh\beta J)^N + (2\sinh\beta J)^N$ "well behaved"

No phase transition in $d=1$

Historically led to people thinking Ising model boring

Only later realized that there $\underline{\underline{\text{is}}}$ a phase transition in $d=2$! $\quad k_B T_c = 2.269\, J$

② Can do Ising model in field in $d=2$ by TM

$$E = -J \sum S_i S_j - B \sum S_i$$

$$Z = \sum_{S_1 S_2 \ldots S_N} e^{\beta J \sum S_i S_j + \beta B \sum S_i}$$

Be careful with counting of B term

$$\underbrace{e^{\beta J S_1 S_2 + \beta B/2 (S_1 + S_2)}}_{\equiv M(S_1, S_2)} \; e^{\beta J (S_2 S_3) + \beta B/2 (S_2 + S_3)}$$

$$M = \begin{array}{c|cc} & + & - \\ \hline + & e^{\beta J + \beta B} & e^{-\beta J} \\ - & e^{-\beta J} & e^{\beta J - \beta B} \end{array}$$

Reason is that it costs so little energy to flip many spins in $d=1$

$d=1$: Energy cost is independ. of # flipped spins

$d=2$ Energy cost increases with # flipped spins

I8 B

$$\left(e^{\beta J+\beta B}-\lambda\right)\left(e^{\beta J-\beta B}-\lambda\right)-e^{-2\beta J}=0$$

$$\lambda^2-\lambda e^{\beta J}\left(e^{\beta B}+e^{-\beta B}\right)+e^{2\beta J}-e^{-2\beta J}=0$$

$$\lambda=\frac{1}{2}\left\{e^{\beta J}\,2\cosh\beta B\pm\sqrt{e^{2\beta J}\,4\cosh^2\beta B-4\left(e^{2\beta J}-e^{-2\beta J}\right)}\right\}$$

(3) spin-spin correlation functions

$$\langle S_1 S_{1+l}\rangle = Z^{-1}\sum_{S_1 S_2 \ldots S_N} S_1 S_{1+l}\, e^{-\beta E}$$

$$= Z^{-1}\sum_{S_1 S_2 \ldots S_N} S_1 e^{\beta J S_1 S_2}\, e^{\beta J S_2 S_3}\cdots S_{1+l}\, e^{\beta J S_{1+l} S_{2+l}}\cdots$$

$\cosh\beta J + S_1 S$