Ising Model

One of most simple examples of interacting statistical mechanics system.

\[ E = -J \sum_{\langle i,j \rangle} s_i s_j \]

"spin" degree of freedom

\[ s_i = \pm 1 \] living on site \( i \)

Illustrates many ideas of interacting stat mech in simple context

(both methods and concepts)

near neighbors

sites on some lattice

Example \( N = 4 \) site linear chain pbc

\[ E = -J (s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_1) \]

\[ \begin{array}{cccc}
  s_1 & s_2 & s_3 & s_4 \\
  + & + & + & + \\
  - & - & - & - \\
  + & + & - & - \\
  + & + & + & - \\
  + & - & + & - \\
  - & - & + & + \\
  + & + & - & + \\
  - & - & - & - \\
  + & - & - & - \\
  - & + & - & - \\
  + & - & + & + \\
  - & + & + & + \\
\end{array} \]

\[ E \]

\[ \begin{array}{c}
  -4J \\
  -4J \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  +4J \\
  +4J \\
\end{array} \]

"Ferromagnetic"

configurations have lowest \( E \)

"antiferromagnetic"

\[ \begin{array}{c}
  + - - + \\
  - + + - \\
\end{array} \]

\[ \begin{array}{c}
  + - + - \\
  - + - + \\
\end{array} \]

have highest energy"
\[ Z = \sum_{S,S_2,S_3,S_4} e^{-\beta E} = 2 e^{4\beta J} + 12 + 2 e^{-4\beta J} \]

Equation:

- Competition between \( \langle E \rangle \) minimization and entropy.
- To minimize \( F = \langle E \rangle - TS \).
  - \( T \) controls balance of priori between \( \langle E \rangle \) and \( S \).
- Picture of phase transition: \( T_c \) is "critical temperature" dividing whether \( \langle E \rangle \) wins or \( S \) wins.

Phase transition: sharp change in behavior.

\[ Z \text{ or a derivative thereof is non-analytic.} \]

How could \( \sum e^{-\beta E} \) be non-analytic?!?!

\[ \text{Sum must be infinite} \]
Simple example

\[ 1 + x + x^2 + x^3 + \ldots + x^n \] is analytic

but infinite sum is \( \frac{1}{1-x} \) which has pole at \( x = 1 \)

\[ <M> \]

\[ <M> \equiv \frac{1}{N} \sum \delta_{i} \]

\[ M \sim (T_c - T)^{\beta} \]

\[ \chi = \frac{d <M>}{dT} \sim (T_c - T)^{\gamma} \]

\[ T \rightarrow T_c \]

\[ T \rightarrow T_c \]
\[ F = - \sum_{i,j} s_i s_j - B \sum_i s_i \]

\[ \langle M \rangle = z^{-1} \sum e^{-\beta E} M \]

\[ = z^{-1} \sum e^{\beta E \sum_i s_i + \beta B \sum_i s_i} M \]

\[ \frac{d\langle M \rangle}{dB} = z^{-1} \sum e^{\beta E \sum_i s_i + \beta B \sum_i s_i} \beta M - \beta M \cdot \frac{1}{z} \]

\[ \frac{d\langle M^2 \rangle}{dB} = \beta \left[ \langle M^2 \rangle - \langle M \rangle^2 \right] \]

N.B. It is very weird that \( \langle M \rangle \) to be zero.

Global Ising model has symmetry \( S_i \rightarrow -S_i \) \( \forall i \)

\( E \rightarrow -E \)

so configurations come in pairs \( \pm M \) same \( E \).

Such spontaneous symmetry breaking is very mysterious!
Connections to HEP

(1) $\phi^4$ field theory

$-\infty < \phi < \infty$ continuous variable

$$E = -J \sum \phi_i \phi_j + \lambda \sum (\phi_i^2 - 1)^2$$

gives Ising-like character

Key issue: Does $\phi$ get "vacuum expectation value" $\langle \phi \rangle \neq 0 \Rightarrow$?

Like magnetic phase transition: $\langle M \rangle \neq 0$ at low $T$.

Note also $\frac{d^4}{dx^4} = \frac{1}{dx} \left[ \phi(x) + dx \frac{d}{dx}(x + dx) \right]$

so $(\frac{d^4}{dx^4})^2 \sim \phi(x + dx) \phi(x) \sim \phi_i \phi_j$

So really QFT looks at $S = \frac{1}{2} \int (d^4 \phi)^2 dx$

$$+ \lambda \int (\phi_i^2 - 1)^2 dx$$

(2) Ising model has global symmetry $S_i \rightarrow -S_i$

"Wagner" Ising model

$E = -J \sum S_i \cdot S_j \cdot S_k \cdot S_l$

$S_i = \pm 1$ still but

$S_i$ live on links of square lattice

pick a vertex $x$ and flip Ising variables on the remaining links.

$E$ is unchanged.
Note Ising is simpler than Heisenberg model we encountered in 215B.

\[ H = + J \frac{1}{4} \sum_{\langle ij \rangle} S_i \cdot S_j \]

More commonly in solid this sign favors AF

\[ H_{\text{spin}} = \sum \frac{1}{2} \left( S_i \cdot S_j + \frac{1}{2} (S_i^+ S_j^- + S_i^- S_j^+) \right) \]

\[ H |++++> = J |++++> \]

\[ H |++--> = \phi |++--> + \frac{1}{2} |++--> + \frac{1}{2} |++--> \]

Mix states most diagonalize

Ising is matrix to get energy eigenvalues just this term
Another critical exponent is $\delta$.

Right on dividing line:

$T > T_c$

Spontaneous symmetry breaking:

$M \to 0$ at $B = 0$.

$T = T_c$

$M \propto B^{1/\delta}$

$X = \frac{dM}{dB}$ diverges

(as noted earlier)

$T < T_c$

It is sort of plausible that $X$ should diverge at $T_c$. $T_c$ is the place where system decides spontaneously to magnetize, one might expect its response to an external field to be very strong if already it wants to order on its own!
We can solve the model in $d = 1$ by the "transfer matrix" method.

$$Z = \sum_{s_1, s_2, \ldots, s_N} e^{\beta J \sum_{i,j} s_i s_j}$$

$$= \sum_{s_1, s_2, \ldots, s_N} e^{\beta J s_1 s_2} e^{\beta J s_2 s_3} \cdots e^{\beta J s_N s_1}$$

Define $M(s, s') = e^{\beta J s s'} \sum_{s''=1}^{s'=1} \left( e^{\beta J} e^{-\beta J} \right)$

$$Z = \sum_{s_1 s_2 \ldots s_N} e^{\beta J s_1 s_2} M(s_2 s_3) \cdots M(s_N s_1)$$

$$= \sum_{s_2} \text{of this gives } M^2(s, s_3)$$

Then

$$\sum_{s_3} M^2(s, s_3) M(s_3 s_4) = M^3(s, s_4)$$

$$Z = \sum_{s_1} M^N(s, s_1) = \text{Tr } M^N$$

$$\text{Tr } M^N = \lambda_1^N + \lambda_2^N$$

$\lambda_1, \lambda_2$ eigenvalues of $-M$

$$= 2 \cos h \beta J, 2 \sin h \beta J$$
\[ Z = (2 \cos h \beta J)^N + (2 \sin h \beta J)^N \]

Well behaved as \( N \to \infty \).

No phase transition in \( d = 1 \) (but there is one in \( d \geq 2 \)).

Check \( N = 4 \):

\[ (e^{\beta J} + e^{-\beta J})^4 + (e^{\beta J} - e^{-\beta J})^4 \]

\[ = e^{4\beta J} + 4 e^{2\beta J} + 6 e^0 + 4 e^{-2\beta J} + e^{-4\beta J} \]

\[ + \quad + \quad + \quad + \]

\[ 2 e^{4\beta J} + 12 e^0 + 2 e^{-4\beta J} \]
More on transfer matrix in $d=1$

$1. \quad Z = (2 \cosh \beta J)^N + (2 \sinh \beta J)^N \quad \text{well behaved}$

No phase transition in $d=1$

Historically led people thinking Ising model boring

Only later realized that there is a phase transition in $d=2$

$\quad k_B T_c = 2.269 J$

Reason is that if costs so little energy to flip many spins, in $d=1$

Can do Ising model in field in $d=2$ by TM:

$\quad E = -J \sum_i S_i S_j - B \sum_i S_i$

$\quad Z = \sum_{S_1 S_2 \ldots S_N} e^{\sum \beta J S_i S_j + \beta B S_i S_j}$

Be careful with counting of states

$\quad \equiv M (S_1 S_2)$

$\quad M = \frac{e^{\beta J} + e^{-\beta J}}{e^{\beta J} - e^{-\beta J} + e^{-\beta J} - e^{\beta J}}$
\[
\lambda^2 - \lambda \left( e^{\beta J} + e^{-\beta J} \right) + e^{\beta J} - e^{-\beta J} = 0
\]

\[
\lambda = \frac{1}{2} \left( e^{\beta J} e^{\beta J} + \sqrt{e^{2\beta J} 4 \cosh^2 \beta B - 4 \left( e^{2\beta J} - e^{-2\beta J} \right)} \right)
\]

3. Spin-spin correlation functions

\[
\langle S_i S_{i+t} \rangle = 2^{-1} \sum_{S_1 S_2 \ldots S_N} S_i S_{i+t} e^{-\beta E}
\]

\[
= 2^{-1} \sum_{S_1 S_2 \ldots S_N} S_i e^{\beta J S_i S_2} e^{\beta J S_2 S_3} \ldots S_{i+t} e^{\beta J S_{i+t} S_t} \ldots e^{\cosh \beta J + S_i S_j}
\]