Sturm Liouville Theory

One of our QM postulates is that $\hat{H}$ is Hermitian. We know then that it has a "complete set of eigenstates ϕ_n. What this assures us is that when we get to the end of a calculation of ϕ_n they are complete, eg for Quantum harmonic oscillator

Q: what are they? Hermite polynomials ← all functions can be expanded in Hermite polynomials.

To me this brings to mind a general question. Given a linear differential operator, how do we know when it is Hermitian

$$\mathcal{L} = p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$$

$$\mathcal{L}\phi = \lambda\phi \qquad \text{eigenfunctions}$$

~~Q: what is def of Hermiticity? For matrix $H_{ji}^* = H_{ij}$~~

A: $\mathcal{L}_{fg} = \langle f|\mathcal{L}|g\rangle = \int_a^b f^*(x)\mathcal{L}g(x) \qquad \mathcal{L}_{fg} = \mathcal{L}_{gf}^*$

Simplest example $\mathcal{L} = d^2/dx^2$

Q. What are eigenfunctions $\phi(x)$?

A: $\sin nx$ and $\cos nx$ with $\lambda = -n^2$

We use completeness of these in Fourier transforms

How do we know this $\mathcal{L}$ is Hermitian?!

Let's assume f, g real for simplicity

$$\mathcal{L}_{fg} = \langle f | \mathcal{L} | g \rangle$$

$$= \int_a^b dx\, f\,(p_0 g'' + p_1 g' + p_2 g)$$

$$\mathcal{L}_{gf} = \int_a^b dx\, g\,(p_0 f'' + p_1 f' + p_2 f)$$

integrate by parts

$$= \int_a^b dx \left[-(g p_0)' f' - (g p_1)' f + p_2 f g \right] + g p_0 f' + g p_1 f \Big|_a^b$$

$$= \int_a^b dx \left[(g p_0)'' f - (g p_1)' f + p_2 f g \right]$$

$$+ g p_0 f' + g p_1 f - (g p_0)' f \Big|_a^b$$

$$(g p_0)'' = (g' p_0 + g p_0')' = g'' p_0 + 2 g_0' p_0' + g p_0''$$

SL-3

$$\mathcal{L}_{fg} = \int_a^b dx\; f(p_0 g'' + p_1 g' + p_2 g)$$

$$\mathcal{L}_{gf} = \int_a^b dx \left[(g'' p_0 + 2g' p_0' + g p_0'') f \;-\; (g' p_1 + g p_1') f + p_2 f g \right.$$

$$\left. + \; g p_0 f' + g p_1 f - (g p_0)' f \right|_a^b$$

Difference/
Unmatched terms: $\underbrace{f(2p_0' - p_1)g' + f(p_0'' - p_1')g}_{\mathcal{L}_{gf}} = \underbrace{f p_1 g'}_{\mathcal{L}_{fg}}$

NEED $\quad 2f(p_0' - p_1)g' + f(p_0'' - p_1')g = 0$

very simple: $\quad p_1 = p_0'$!

What about "surface" or "bdy" term

$$g p_0 f' + g p_1 f - g' p_0 f - g p_0' f$$

combine to zero

$$p_0 (g f' - g' f) \Big|_a^b = 0$$

1. Dirichlet: $\quad f, g$ vanish at a, b
2. Neumann: $\quad f', g'$ " " "
3. PBC $\quad f(a) = f(b)$; $f'(a) = f'(b)$ similarly for g

Very simple bottom line

$$\mathcal{L} = p_0 \frac{d^2}{dx^2} + p_1 \frac{d}{dx} + p_2$$

is Hermitian if $p_1 = p_0'$ + BC

$$\mathcal{L} = \frac{d}{dx}\left(p_0 \frac{d}{dx}\right) + p_2$$ is sometimes written

Fourier $p_0 = 1$ $p_1 = p_2 = 0$ ✓✓

Legendre $\mathcal{L} = \underbrace{(1-x^2)}_{p_0}\frac{d^2}{dx^2} \underbrace{-2x}_{p_1}\frac{d}{dx} + \underbrace{n(n+1)}_{p_2}$ ✓✓

Actually there is a more general statement. Any 2nd order linear differential operator can be made Hermitian, even if it doesn't obey $p_1 = p_0'$ with an appropriate "weight function". See Math Phys course...