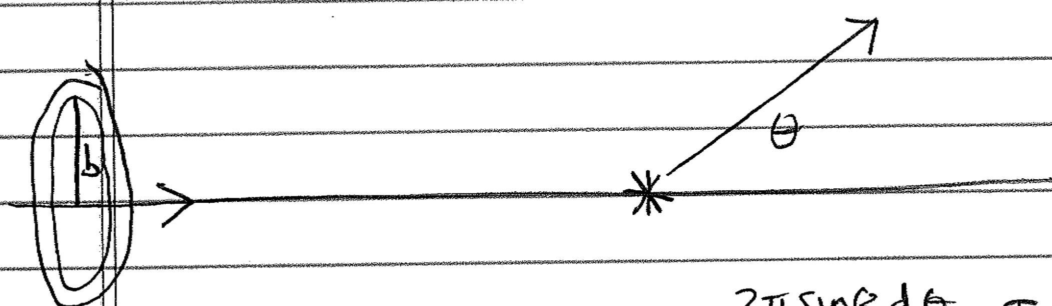
REV -1

Recall scattering theory definitions

$$d\sigma = \sigma(\Omega)\, d\Omega \equiv \frac{\#\text{ particles scattered into solid angle } d\Omega \text{ per unit time}}{\text{incident intensity}}$$

In terms of "impact parameter" b



· Most often σ depends only on θ (azimuthal symmetry)

$$I\, 2\pi b\, db \longleftrightarrow \underbrace{2\pi \sin\theta\, d\theta}_{d\Omega}\ \sigma(\theta)\, I$$

$$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

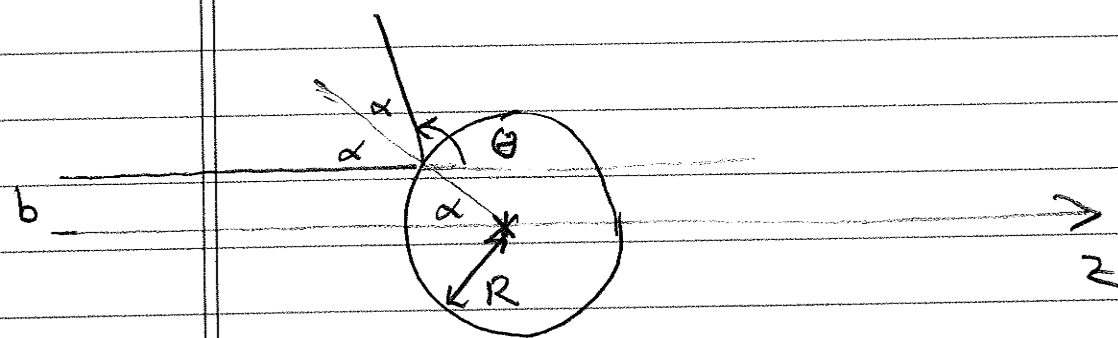
We did one problem: very fast incoming particle going past $1/r$ potential.

"Total cross section"

$$\sigma = \int \sigma(\theta)\, d\Omega$$

REV-2

Another famous scattering problem is hard sphere



$\theta = 0$ if $b > R$

$\sin\alpha = b/R \qquad \theta = \pi - 2\alpha \qquad \alpha = \frac{\pi}{2} - \frac{\theta}{2}$

$$b = R\sin\alpha = R\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$b = R\cos\theta/2$$

check limits: $b = 0 \quad \theta = \pi$

$b = R \quad \theta = 0$

✓

$$\sigma(\theta) = \frac{b}{\sin\theta}\, db/d\theta$$

$$= \frac{R\cos\theta/2}{\sin\theta}\left|-R\tfrac{1}{2}\sin\theta/2\right|$$

$$= \frac{1}{4}R^2$$ ← indep of θ (not typical)

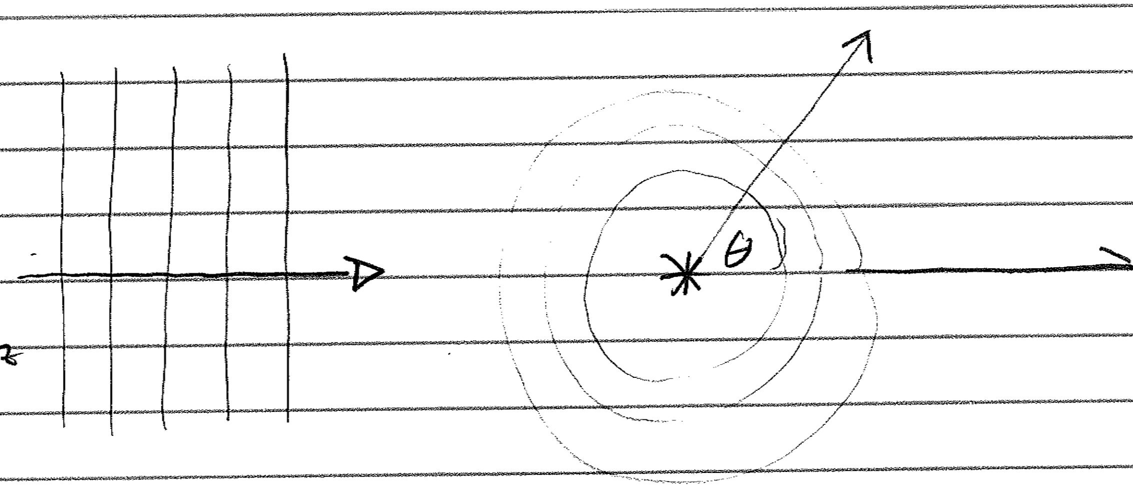
$$\sigma = \int \sigma(\theta)\, d\Omega = \int \tfrac{1}{4}R^2\, 2\pi\sin\theta\, d\theta$$

$$= \frac{1}{4}R^2\, 4\pi = \pi R^2$$ ✓✓

In QM imagine plane wave incoming instead of particle (because plane wave is soln of Sch: eqn) then hits scattering center. Look for solution of form

"scattering amplitude"

$$\psi(\vec{r}) = A\left\{ e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \right\}$$



$k = \sqrt{2mE}/\hbar$

probability density

Incident flux is $|A|^2 \; v\,dt\, d\sigma$ (usual $I = nevA$ argument)

Scattered flux is $\dfrac{|A|^2 |f|^2}{r^2} \; v\,dt\; r^2 d\Omega$

Setting these equal $\sigma(\theta) = \dfrac{d\sigma}{d\Omega} = |f|^2$

Sc-2

Schroedinger Eqn in spherical potential $[\hat{H}, \hat{L}^2] = [\hat{H}, \hat{L}_z] = 0$

$$\psi(r\theta\phi) = R(r) Y_{\ell m}(\theta,\phi)$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}u + \left\{V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right\} u = E u$$

where $u(r) = R(r)\, r$

Very large r $\quad V(r) \to 0$ as does $\frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}$

$$\frac{d^2 u}{dr^2} = -k^2 u$$

$$u(r) = C e^{ikr} + D e^{-ikr}$$

outgoing

$$R(r) = e^{ikr}/r$$

further motivating our guess of form for solution

Assume $V(r)$ is localized, ie there is a region where $V(r)$ might be big, but also an intermediate region where $V(r)$ can be ignored but $\frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}$ cannot

might seem problematic
eg Coulomb $1/r$ but
mostly screened Yukawa $e^{-\alpha r}/r$

SC-3

Q: Does anyone recall solns to Radial Eqn for $V=0$?

A: $R(r) = A\, j_\ell(kr) + B\, \eta_\ell(kr)$

Bessel functions.

We emphasized Bessel a lot like sin, cos

except they decay and have unevenly spaced roots.

Another similarity $\qquad \sin\theta, \cos\theta \leftrightarrow e^{\pm i\theta}$

$$e^{i\theta} = \sin\theta + i\cos\theta$$

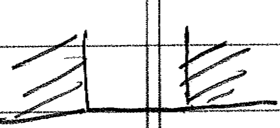
"Hankel functions" $\qquad h_\ell^1 \equiv j_\ell + i\eta_\ell$

$$h_\ell^2 \equiv j_\ell - i\eta_\ell$$

$h_\ell^1 \sim e^{ikr}/r \qquad h_\ell^2 \sim e^{-ikr}/r$

$$\psi(r,\theta,\phi) = A\left[e^{ikz} + \sum_{\ell m} C_{\ell m}\, h_\ell^1(kr)\, Y_{\ell m}(\theta,\phi)\right]$$

Analogy: particle in a box $\qquad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$



0 $\quad$ a

$\psi(x) = e^{\pm ikx} \qquad E = \frac{\hbar^2 k^2}{2m}$

But choose $\sin kx$ / linear combo to satisfy Boundary condition

SC-3A

Spherical Hankel functions

$$h_0^1 = -i\,\frac{e^{ix}}{x} \qquad\qquad h_0^2 = i\,\frac{e^{-ix}}{x}$$

$$h_1^1 = \left(-\frac{i}{x^2} - \frac{1}{x}\right)e^{ix} \qquad\qquad h_1^2 = \left(\frac{i}{x^2} - \frac{1}{x}\right)e^{-ix}$$

$$h_2^1 = \left(-\frac{3i}{x^2} - \frac{3}{x^2} + \frac{i}{x}\right)e^{ix} \qquad\qquad h_2^2 = \left(\frac{3i}{x^2} - \frac{3}{x^2} + \frac{i}{x}\right)e^{-ix}$$

$$j_0 = \frac{\sin x}{x} \qquad\qquad \eta_0 = \frac{-\cos x}{x}$$

$$j_1 = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad\qquad \eta_1 = \frac{-\cos x}{x^2} - \frac{\sin x}{x}$$

etc

$$\underbrace{j_\ell \sim x^\ell \frac{2^\ell \ell!}{(2\ell+1)!} \qquad\qquad \eta_\ell \sim \frac{1}{x^{\ell+1}}\left(-\frac{(2\ell)!}{2^\ell \ell!}\right)}_{x \text{ small}}$$

Sc-4

$Y_{\ell m}(\theta,\phi) \sim P_\ell(\cos\theta)$ for $m=0$

Spherically symmetric $\rightarrow$ no ϕ dependence

$$\psi(r,\theta) = A\left[e^{ikz} + k\sum_{\ell=0}^{\infty} i^{\ell+1}(2\ell+1)\, a_\ell\, h_\ell^1(kr) P_\ell(\cos\theta)\right]$$

We said $h_\ell^1(kr) \sim \frac{e^{ikr}}{r}$ large r

In fact $h_\ell^1(kr) = \frac{e^{ikr}}{kr}(-i)^{\ell+1}$ large r

These various factors just a question of normalization of a_ℓ. We will see why this is convenient

very large r

$$\psi(r,\theta) = A\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right]$$

$$f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\, a_\ell\, P_\ell(\cos\theta)$$

"partial wave amplitude"

Total cross section

$$\sigma = \int |f|^2\, d\Omega$$

$$= \int \sum_\ell \sum_{\ell'} (2\ell+1)(2\ell'+1)\, a_\ell\, a_{\ell'}\, P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\, d\Omega$$

$$= 4\pi \sum_{\ell=0}^{\infty}(2\ell+1)|a_\ell|^2$$

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

It is better to write everything ie including incoming wave in same formalism. Recall

$$e^{ikz} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta)$$

$$\psi(r,\theta) = A \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) \left[j_\ell(kr) + ik a_\ell h_\ell^1(kr) \right] P_\ell(\cos\theta)$$

Strategy for scattering problems:

① Solve Schroedinger eqn where $V(r) \neq 0$ to determine coefficients a_ℓ

② $a_\ell \to f(\theta) \to \sigma(\theta)$

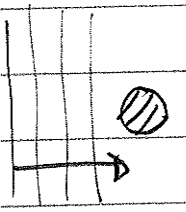
$a_\ell \to \sigma$

SC-6

Ex: Hard sphere scattering

HW 5-4 is finite depth problem.

$$V(r) = \begin{cases} \infty & r \le a \\ 0 & r > a \end{cases}$$



Q: Any guesses concerning total σ?

$$\psi(a,\theta) = 0$$

wave incoming not a particle!

$$\sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1)\left[j_\ell(ka) + ik a_\ell h_\ell^1(ka)\right] P_\ell(\cos\theta) = 0$$

Since P_ℓ are indep, the individual coefficients must vanish

$$a_\ell = i\, j_\ell(ka) / k h_\ell^1(ka)$$

$$\sigma = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left| j_\ell(ka) / h_\ell^1(ka) \right|^2$$

Not much intuition here because who understands j_ℓ and h_ℓ^1 ?!?!

Consider low energy scattering $ka \ll 1$ HW (like 5-4)

$$\frac{j_\ell(z)}{h_\ell^1(z)} = \frac{j_\ell(z)}{j_\ell(z) + i\eta_\ell(z)} \underset{z \text{ small}}{\approx} -i\frac{j_\ell(z)}{\eta_\ell(z)}$$

SC-7

$$\frac{j_\ell(ka)}{h_\ell'(ka)} \sim \frac{i}{2\ell+1}\left[\frac{2^\ell \ell!}{(2\ell)!}\right]^2 (ka)^{2\ell+1}$$

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)}\left[\frac{2^\ell \ell!}{(2\ell)!}\right]^4 (ka)^{4\ell+2}$$

$ka \ll 1$ keep only $\ell = 0$

$$\sigma = 4\pi a^2$$

four times classical answer

"diffraction" of wave.

"Born Approximation" is an alternate approach to scattering theory which is perturbative in V
~~Starting point is to recast~~ Sch Eqn as integral eqn.

Another Example of diff eqn $\Leftrightarrow$ integral eqn

is provided by the starting point of scattering theory

Consider - Sch. Eqn in position representation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r)+V(r)\psi(r)=E\psi(r)$$

Work instead in momentum space

$$\psi(r)=\frac{1}{(2\pi)^{3/2}}\int\psi(k_1)e^{-ik_1r}d^3k_1$$

$$\frac{1}{(2\pi)^{3/2}}\int d^3k_1\frac{\hbar^2k_1^2}{2m}\psi(k_1)e^{-ik_1r}+V(r)\int\frac{d^3k_1}{(2\pi)^{3/2}}e^{-ik_1r}\psi(k_1)$$

$$=E\int\frac{d^3k_1}{(2\pi)^{3/2}}e^{-ik_1r}\psi(k_1)$$

Multiply by e^{ikr} and $\int d^3r/(2\pi)^3$

Using identity $\int\frac{d^3r}{(2\pi)^3}e^{i(k-k_1)r}=\delta(k-k_1)$

5C-9

$$\frac{\hbar^2 k^2}{2m}\psi(k) + \underbrace{\frac{1}{(2\pi)^3}\int d^3r\, V(r)\, e^{ikr}\int d^3k_1 e^{-ik_1 r}\psi(k_1)}_{\int d^3k_1 \frac{1}{(2\pi)^3}\int d^3r\, e^{i(k-k_1)r} V(r)\, \psi(k_1)} = E\psi(k)$$

$$\int d^3r\, e^{i(k-k_1)r} V(r) \equiv V(k-k_1)$$

$$\frac{\hbar^2 k^2}{2m}\psi(k) + \int d^3k_1\, V(k-k_1)\psi(k_1) = E\psi(k)$$

$E(k)$

A physically pleasing eqn $\psi(k_1)$ gets linked to $\psi(k)$ by Fourier component of potential V at $k-k_1$!

$$\left(E(k) - E\right)\psi(k) = -\int d^3k_1\, V(k-k_1)\psi(k_1)$$

Can imagine trying to solve this in a similar way to our derivation of time evolution operator...

Although this integral Eqn for $\psi(k)$ is enlightening, it is often most simple to work in real space. The analogous formula is

$$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|r-r_0|}}{|r-r_0|} V(r_0)\psi(r_0)d^3r_0$$

soln to free particle Sch. Eqn ($V=0$)

In words: soln for ψ at $\vec{r}$ depends on value at r_0 times a "propogator" from r_0 to r, and integrated over all space d^3r_0

This formula is very useful because it allows an iterative (perturbative) soln.

$$\psi(r) = Ae^{ikz} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\vec{k}\cdot\vec{r}_0} V(r_0)\psi(r_0)d^3r_0$$

using $\frac{1}{|r-r_0|} \sim \frac{1}{r}$ and $e^{ik|\vec{r}-\vec{r}_0|} \approx e^{ikr}e^{-i\vec{k}\cdot\vec{r}_0}$

$$\Rightarrow \quad f(\theta,\phi) = -\frac{m}{2\pi\hbar^2 A}\int e^{-i\vec{k}\cdot\vec{r}_0}\, V(r_0)\, \psi(r_0)\, d^3r_0$$

Born approx replaces this by $A e^{ikz_0}$

$= A e^{i\vec{k}'\cdot\vec{r}_0}$

with $k' = k\hat{z}$

$$\boxed{f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i(k'-k)\cdot r_0}\, V(r_0)\, d^3r_0}$$