$\boxed{\text{Scattering Theory Summary}}$

We have discussed 3 approaches

$$\psi(r,\theta) = A\left[e^{ikz} + f(\theta)\, e^{ikr}/r\right]$$

(1) $$k \sum_{\ell=0}^{\infty} i^{\ell+1}(2\ell+1)\, a_\ell\, h_\ell^{(1)}(kr)\, P_\ell(\cos\theta)$$

$$f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, a_\ell\, P_\ell(\cos\theta)$$

$$\sigma = 4\pi \sum_{\ell=0}^{\infty} (2\ell+1)\,|a_\ell|^2$$

$\therefore$ goal is computation of a_ℓ

Our example was hard sphere scattering where

a_ℓ determined by requiring $\psi(r=a,\theta) = 0$

(2) $$\psi(r,\theta) = A \sum_\ell \frac{2\ell+1}{2ikr}\left[e^{ikr+2\delta_\ell} - (-1)^\ell e^{-ikr}\right] P_\ell(\cos\theta)$$

$$f(\theta) = \frac{1}{k} \sum (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell\, P_\ell(\cos\theta)$$

$$\sigma = \frac{4\pi}{k^2} \sum (2\ell+1) \sin^2\delta_\ell$$

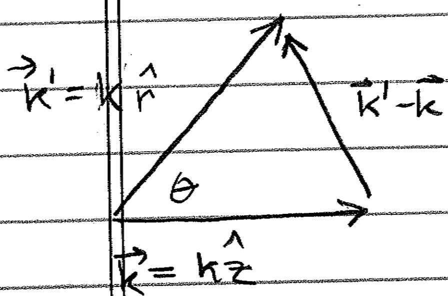
~~low E: focus on δ_0~~ $\leftarrow$ $\boxed{\text{PS-4A}}$

Our example ~~here~~ was $V(r) = A/r^2$ $\leftarrow$ $\boxed{\text{PS-5}}$

scs - 2

③ Born approximation

$$f(\theta) = \frac{-m}{2\pi\hbar^2} \int e^{i(\vec{k}'-\vec{k})\cdot\vec{r}'} V(\vec{r}')\, d^3r'$$



$$\kappa = |\vec{k}'-\vec{k}| = 2k\sin\theta/2$$

Elastic $|\vec{k}| = |\vec{k}'|$

An impt special case is low $E = \hbar^2k^2/2m$

k small $\quad e^{i(\vec{k}'-\vec{k})\cdot\vec{r}} \to 1$

$$f(\theta) = \frac{-m}{2\pi\hbar^2} \int V(r')\, d^3r'$$

Spherically symmetric potential: $\quad V(\vec{r}') = V(r')$

just $|\vec{r}'|$

Choose $\hat{z}'$ axis along $\vec{\kappa}$ to do integral

$$\vec{\kappa}\cdot\vec{r}' = \kappa r'\cos\theta'$$

$$f(\theta) = \frac{-m}{2\pi\hbar^2} \int e^{i\kappa r'\cos\theta'} V(r')\, r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi'$$

$$= \frac{-m}{\hbar^2} \int V(r') r'^2 dr' \underbrace{\int_0^\pi e^{i\kappa r'\cos\theta'} \sin\theta'\, d\theta'}_{-e^{i\kappa r'\cos\theta'}/i\kappa r' \Big|_0^\pi \;=\; \frac{2\sin\kappa r'}{\kappa r'}}$$

$$\boxed{f(\theta) = \frac{-2m}{\hbar^2\kappa} \int_0^\infty r' V(r') \sin\kappa r'\, dr' \qquad \kappa = 2k\sin\theta/2}$$

Some Examples of Born approximation

① Low Energy soft sphere scattering

$$V(r) = \begin{cases} V_0 & r \le a \\ 0 & r > a \end{cases}$$

$$f(\theta) = \frac{-m}{2\pi\hbar^2} \int V(r') d^3r'$$

$$= \frac{-m}{2\pi\hbar^2} \frac{4}{3}\pi a^3 V_0$$ ← indep of θ

Q: reasonable?

if particle comes in very fast expect $f(\theta)$ peak at $\theta = 0$

$$\sigma = \int d\Omega |f|^2$$

$$= 4\pi \left(\frac{2mV_0 a^3}{3\hbar^2} \right)^2$$

② Yukawa potential

$$V(r) = \beta \, e^{-\mu r}/r$$

$$f(\theta) = \frac{-2m\beta}{\hbar^2 \kappa} \int_0^\infty e^{-\mu r'} \sin\kappa r' \, dr'$$

$$= \frac{-2m\beta}{\hbar^2\kappa} \frac{1}{2i} \int_0^\infty \left[e^{(-\mu+i\kappa)r'} - e^{(-\mu-i\kappa)r'} \right] dr'$$

$$+ \left[\frac{e^{(-\mu+ik)r'}}{-\mu+ik} - \frac{e^{(-\mu-ik)r'}}{-\mu-ik} \right]_0^\infty$$

$$- \left[\frac{-\mu-ik+\mu-ik}{\mu^2+k^2} \right]$$

SCS - 4

$$f(\theta) = \frac{-m\beta}{\hbar^2 \kappa} \frac{1}{i} \frac{2i\kappa}{\mu^2 + \kappa^2} = \frac{-2m\beta}{\hbar^2(\mu^2+\kappa^2)}$$

Recall $\kappa = 2k \sin \theta/2$

$$f(\theta) = -\frac{2m\beta}{\hbar^2} \frac{1}{\mu^2 + 4k^2 \sin^2\frac{\theta}{2}}$$

forward scattering $\theta = 0$ $f(0) = -\frac{2m\beta}{\hbar^2} \frac{1}{\mu^2}$

back scattering $\theta = \pi$ $f(\pi) = -\frac{2m\beta}{\hbar^2} \frac{1}{\mu^2 + 4k^2}$

$f(\pi) \ll f(0)$ if $k \gg \mu$ (high energy)

$f(\pi) \sim f(0)$ if $k \ll \mu$ (low energy)

Some Born problems you could try:

$V(r) = V_0 e^{-\mu^2 r^2}$

$V(r) = V_0 e^{-\mu r}$

$V(r) = B\,\delta(r)$

SCS - 5

Total cross section for Yukawa

$$\sigma = \int |f|^2 \, d\Omega$$

$$= \left(\frac{2m\beta}{\hbar^2}\right)^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \left(\mu^2 + 4k^2 \sin^2\frac{\theta}{2}\right)^{-2}$$

$2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

let $x = \sin\theta/2$

$dx = \frac{1}{2}\cos\theta/2 \, d\theta$

θ	x
0	0
π	1

$$\int 2x \, 2dx \left(\mu^2 + 4k^2 x^2\right)^{-2}$$

$$\rightarrow \frac{-2}{4k^2} \left(\mu^2 + 4k^2 x^2\right)^{-1} \Big|_0^1$$

$$\rightarrow -\frac{1}{2k^2}\left(\mu^2 + 4k^2\right)^{-1} + \frac{1}{2k^2}\frac{1}{\mu^2}$$

$$\rightarrow \frac{1}{2k^2}\left(\frac{1}{\mu^2} - \frac{1}{\mu^2 + 4k^2}\right) = \frac{1}{2k^2} \frac{1}{\mu^2(\mu^2 + 4k^2)} \quad \left[\mu^2 + 4k^2 - \mu^2\right]$$

$$\rightarrow \frac{2}{\mu^2(\mu^2 + k^2)}$$

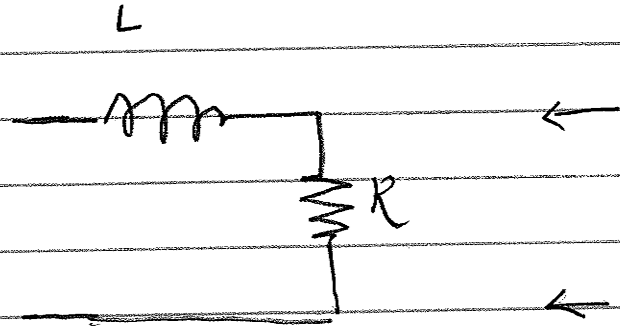
$$\sigma = \left(\frac{2m\beta}{\hbar^2}\right)^2 2\pi \, 2 \Big/ \mu^2(\mu^2 + k^2)$$

C-1

"Scattering and Phase Shifts in Circuits"

"Incoming voltage

$V_0 e^{i\omega t}$



outgoing voltage along resistor

$$V_0 e^{i\omega t} - L\frac{dI}{dt} - IR = 0$$

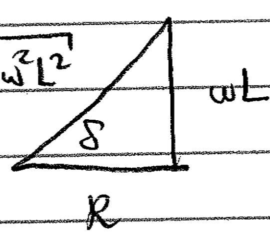
$$I = I_0 e^{i\omega t}$$

$$V_0 - L\,i\omega I_0 - I_0 R = 0$$

$$I_0 = V_0/(R_0 + i\omega L) = \frac{V_0}{Z} e^{-i\delta}$$

$Z = \sqrt{R^2+\omega^2L^2}$

"impedance"



Inductor L produces phase shift

$\tan\delta = \omega L/R$

"outgoing"
"scattered"
voltage

$$I.R = \frac{V_0}{Z} \underbrace{e^{-i\delta} e^{i\omega t}}_{e^{i(\omega t - \delta)}} R$$