Phase Shifts

A convenient way to rewrite a_ℓ and hence reformulate scattering theory.

Useful analogy

Consider a pulse $f(x) = A e^{-(x+x_0)^2/2\sigma^2}$ $\quad x_0 > 0$

gaussian centered at $x = -x_0$

Can Fourier transform $\quad f(x) = \int dk\, a(k)\, e^{ikx}$

($a(k)$ will also be Gaussian! width σ^{-2})

Can make pulse move $\quad \omega = kv$

$$f(x-vt) = \int dk\, a(k)\, e^{ik(x-vt)} = \int dk\, a(k)\, e^{ikx - i\omega t}$$

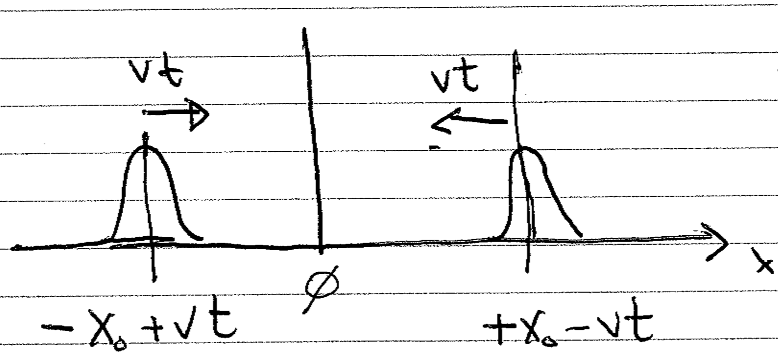
Can examine scattered wave $\quad ikx \to -ikx \qquad$ Peak at $x = -x_0 + vt$

$$\int dk\, a(k)\, e^{-ikx - i\omega t} = f(-x-vt)$$

$$= A e^{-(-x+x_0-vt)^2/2\sigma^2}$$

$$= A e^{-(x-x_0+vt)^2/2\sigma^2}$$

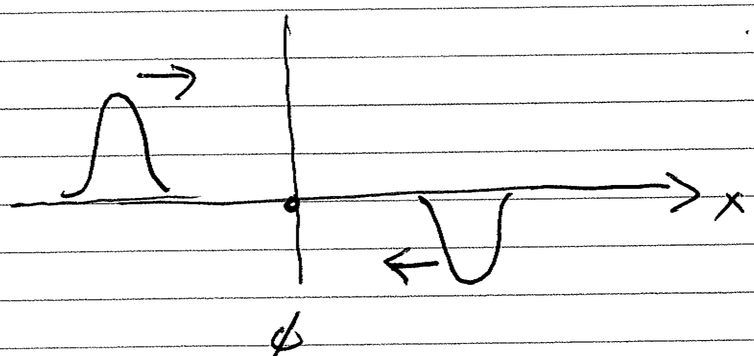
peak at $x = x_0 - vt$



PS-2

Q: Is this what happens when one reflects off a boundary $y(x=0,t)\equiv 0$?

A: No pulse is inverted!



Superposition principle guarantees $y(x=0,t)=0$

So correct scattering answer is

$$\int dk\, e^{-ikx-i\omega t+i\pi} = -Ae^{-(x-x_0+vt)^2/2\sigma^2}$$

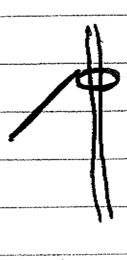
phase shift

$e^{i\pi}=-1$ $\qquad$ $e^{i\delta}$ with $\delta=\pi$

Scattering process is described by phase shift

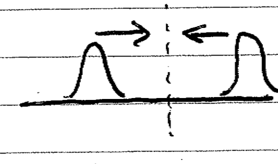
Q: Can you get scattering with $\delta=0$?

A: Yes: Massless/frictionless ring



$$F_y = T\,dy/dx = ma_y = 0 \Rightarrow dy/dx = 0$$

$\Rightarrow$ image wave <u>not</u> inverted



Now do QM problem!

Incoming wave

$$\psi_0 = A e^{ikz} = A \sum_\ell i^\ell (2\ell+1)\, j_\ell(kr)\, P_\ell(\cos\theta)$$

$V=0$

$$= A \sum_\ell i^\ell (2\ell+1) \frac{1}{2}\left[h_\ell^{(1)}(kr) + h_\ell^{(2)}(kr)\right] P_\ell(\cos\theta)$$

kr large $\quad \frac{1}{kr}(-i)^{\ell+1} e^{ikr} \qquad \frac{1}{kr} i^{\ell+1} e^{-ikr}$

$$= A \sum_\ell \frac{(2\ell+1)}{2ikr}\left[e^{ikr} - (-1)^\ell e^{-ikr}\right] P_\ell(\cos\theta)$$

Potential $V(r)$ introduces phase shift

Incoming, unchanged by $V(r)$

$$e^{i(kr+2\delta_\ell)}$$

factor of 2 is a convention

~~Our previous expression~~

$$\psi_{V\neq 0} = A \sum_\ell \frac{(2\ell+1)}{2ikr}\left[e^{i(kr+2\delta_\ell)} - (-1)^\ell e^{-ikr}\right] P_\ell(\cos\theta)$$

Contrast with previous expression

$$\psi_{V\neq 0} = A\left[e^{ikz} + k \sum_{\ell=0}^{\infty} i^{\ell+1}(2\ell+1)\, a_\ell\, h_\ell^{(1)}(kr)\, P_\ell(\cos\theta)\right]$$

which at large r looks like

$$\psi_{V\neq 0} = A \sum_{\ell}\left[\underbrace{\frac{(2\ell+1)}{2ikr}\left(e^{ikr}-(-1)^{\ell}e^{-ikr}\right)}_{e^{ikz}\text{ part}} + \underbrace{\frac{(2\ell+1)}{r}a_{\ell}e^{ikr}}_{h_{\ell}^{(1)}\text{ part}}\right]P_{\ell}(\cos\theta)$$

Clearly $\underbrace{\frac{1}{2ikr}e^{2i\delta_{\ell}}}_{\text{new}} = \underbrace{\frac{1}{2ikr} + \frac{a_{\ell}}{r}}_{\text{previous}}$

coefficients of e^{ikr} term

ie $\quad a_{\ell} = \frac{1}{2ik}\left(e^{2i\delta_{\ell}}-1\right) = \frac{1}{k}e^{i\delta_{\ell}}\sin\delta_{\ell}$

So $\quad f(\theta) = \sum_{\ell}(2\ell+1)\,a_{\ell}\,P_{\ell}(\cos\theta)$

$$= \frac{1}{k}\sum_{\ell}(2\ell+1)\,e^{i\delta_{\ell}}\sin\delta_{\ell}\,P_{\ell}(\cos\theta)$$

$$\sigma = 4\pi\sum_{\ell}(2\ell+1)|a_{\ell}|^2$$

$$= \frac{4\pi}{k^2}\sum_{\ell}(2\ell+1)\sin^2\delta_{\ell}$$

~~at high ene~~

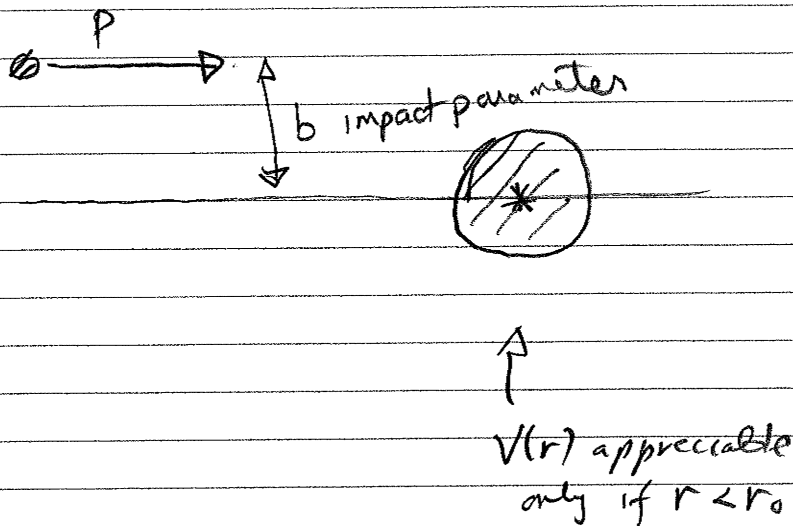
at low energies only $\ell=0$ term impt $\quad \frac{4\pi}{k^2}\sin^2\delta_0$

This is <u>very</u> useful since Bessel eqn much simpler if $\ell=0$!

PS-4A

Why can we focus on δ_0 only at low E ?

Consider classical scattering



Angular momentum

$$L = pb$$

Expect significant scattering only if $b < r_0$

ie $\quad L < p r_0$

Now in QM $\quad L = \hbar\sqrt{\ell(\ell+1)} \sim \hbar \ell$

$$p = \hbar k$$

So we expect significant scattering only for

$$\hbar \ell < \hbar k r_0$$

$$\ell < k r_0 \qquad E = \hbar^2 k^2 / 2m$$

For low E $\;k$ will be small and $k r_0 \ll 1$

Only $\ell = 0$ contributes!

As an example, let's compute phase shifts for $V(r) = \frac{A}{r^2}$

The radial eqn is

$$\left[\frac{d^2}{dr^2} + k^2 - \left(\frac{2mA}{\hbar^2 r^2} + \frac{\ell(\ell+1)}{r^2}\right)\right] R_{\ell k}(r) = \phi$$

Recall that for $A=0$ (free particles) solns are

$$R_{\ell,k}(r) = r\, j_\ell(kr)$$

(Eqn for $u = R/r$ is spherical Bessel Eqn)

Q: Ideas?

A: Define $\lambda(\lambda+1) = \ell(\ell+1) + 2mA/\hbar^2$

$$R_{\ell,k}(r) = r\, j_\lambda(kr) \quad !$$

$$\lambda^2 + \lambda - \ell(\ell+1) + \frac{2mA}{\hbar^2} = \phi$$

$$\lambda = -\frac{1}{2} + \sqrt{(\ell+1/2)^2 + 2mA/\hbar^2}$$

For large x

$j_\ell(x) \sim \frac{1}{x}\sin\left(x - \frac{\ell\pi}{2}\right)$

$$\rightarrow R_{\ell,k}(r) \sim \frac{1}{k}\sin\left(kr - \frac{\pi}{2}\lambda\right)$$

$$\frac{\pi}{2}\lambda = \frac{\pi}{2}\ell + \delta_\ell)$$

$$R_{\ell,k}(r) \sim \frac{1}{k}\sin\left(kr - \frac{\ell\pi}{2} - \delta_\ell\right)$$

So $$\delta_\ell = \frac{\pi}{2}(\lambda - \ell) = \frac{\pi}{2}\left\{\sqrt{(\ell+\tfrac{1}{2})^2 + \frac{2mA}{\hbar^2}} - (\ell+\tfrac{1}{2})\right\}$$

This is an atypical problem in that δ_ℓ is independent of k, Could continue this problem to get

$$f(\theta) = \frac{1}{k}\sum_{\ell} (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell \, P_\ell(\cos\theta)$$

$$\sigma = \frac{4\pi}{k^2} \sum (2\ell+1) \sin^2\delta_\ell$$