# $\boxed{\text{Lifetimes of Excited State}}$

One of most impt examples/applications of

TDPT / Fermi's golden rule is to atomic lifetimes

$$w_{e \to k} = \frac{2\pi}{\hbar} |\langle k | \vartheta | e \rangle|^2 g(E_k)$$

$$\vartheta = \frac{-e}{2mc} (\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p})$$

First discuss $g(E_k) = \frac{\#\ \text{states}}{\text{Energy window}}$

Here photons can come off in any direction

$$g = \frac{k^2\, dk\, d\Omega}{(2\pi)^3\, d(\hbar\omega)} = \frac{\omega^2}{(2\pi)^3} \frac{d\Omega}{\hbar c^3}$$

Using $\omega = ck$ $\quad V$✓

L2

We will finally use Sakurai!

Classically the vector potential obeys wave eqn

$$\nabla^2 \vec{A} - 1/c^2 \, \partial^2 \vec{A}/\partial t^2 = 0$$

$\vec{\nabla} \cdot \vec{A} = 0$
$\vec{B} = \vec{\nabla} \times \vec{A}$
$\vec{E} = -1/c \, \partial \vec{A}/\partial t$

and hence is expressible as Fourier series

$$\vec{A}(\vec{r} t) = \sum_{k\alpha} \left[ a_{k\alpha} \vec{\epsilon}_\alpha e^{i(\vec{k}\cdot\vec{r} - \omega t)} + a^*_{k\alpha} \vec{\epsilon}_\alpha e^{-i(\vec{k}\cdot\vec{r} - \omega t)} \right]$$

Fourier coefficients determine strength of $\vec{E}, \vec{B}$

polarization vector (two of them $\perp \hat{k}$)

Just as for classical oscillator

$$E = \frac{1}{2m} p^2 + 1/2 \, m\omega^2 x^2$$

$E, x, p$ are numbers

$\xrightarrow{QM}$ $\hat{x}, \hat{p}$ operators

or $\hbar\omega(\hat{a}^+ \hat{a} + 1/2)$ $a, \hat{a}$ operators

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \, (\hat{a} + \hat{a}^+)$$

dimensionless

units of length

So too for the oscillators of which the EM field is composed. Thus Quantum mechanically

$$\vec{A}(\vec{r},t) = \sum_{k\alpha} c\sqrt{\frac{\hbar}{2\omega}}\left(\hat{a}_{k\alpha}\vec{\epsilon}_\alpha e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \hat{a}^{+}_{k\alpha}\vec{\epsilon}_\alpha e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right)$$

give $\vec{A}$ correct units for dimensionless $\hat{a}$ operators

Now operators (dimensionless)

We will use electric dipole approximation $e^{\pm i\vec{k}\cdot\vec{r}} \simeq 1$ and notice $e^{\pm i\omega t}$ factors are already accounted for in derivation of Fermi's golden Rule.

The $\hat{a}^{+}_{k\alpha}$ term is the one which can create the photon that is emitted when atom decays.

Putting things together...

transition rate

$$\omega = \underbrace{\frac{2\pi}{\hbar}}_{\text{FGR}}\underbrace{\left(\frac{e}{2mc}\right)^2}_{\frac{e}{2mc}\vec{p}\cdot\vec{A}}\underbrace{\left(c\sqrt{\frac{\hbar}{2\omega}}\right)^2}_{\text{units of } A}\underbrace{\frac{\omega^2}{(2\pi)^3\hbar c^3}}_{\text{DOS}}\, d\Omega\, |\langle k|\vec{p}\cdot\hat{\epsilon}|\ell\rangle|^2$$

$$= \frac{e^2\omega}{8\pi^2 m^2\hbar c^3}\,|\langle k|\vec{p}|\ell\rangle\cdot\hat{\epsilon}|^2\, d\Omega$$

We discussed matrix element of $\vec{p}$ earlier when deriving selection rules $\hat{p} = i\frac{m}{\hbar}[\hat{H}_0, \hat{r}]$

So get $\langle k|\hat{p}|\ell\rangle = \frac{im(E_k - E_\ell)}{\hbar}\langle k|\vec{r}|\ell\rangle$

$$= im\omega\langle k|\vec{r}|\ell\rangle$$

photon energy is $(E_k - E_\ell)/\omega$

There are some details of factors of 2, $\pi$ etc from summing over polarization directions etc. See Sakurai

The final result is

$$\omega = \left(\frac{e^2}{4\pi\hbar c}\right)\frac{4}{3}\frac{\omega^3}{c^2}|\langle k|\vec{r}|\ell\rangle|^2$$

Fine structure constant $^1/_{137}$    correct units?

I am only interested in an order of magnitude estimate, so I will just set $\langle k|\vec{r}|\ell\rangle \sim a_0$ the Bohr radius.

$$w = \frac{1}{137}\ \frac{4}{3}\ \frac{(1.6\cdot 10^{16})^3}{(3\cdot 10^8)^2}\ (.5\cdot 10^{-10})^2$$

$$\sim \underbrace{\frac{1}{1.37}\ \frac{4}{.3}\ \frac{(1.6)^3}{9}\ \frac{1}{4}}_{1/10}\ \frac{10^{-2}\,10^{48}\,10^{-20}}{10^{16}} \sim 10^{9}$$

$c = 3\cdot 10^8$ m/s

$a_0 = .5\cdot 10^{-10}$ m

$\omega = 10\,\text{eV}/\hbar = 10(1.6\cdot 10^{-19})/1.05\cdot 10^{-34} \sim 1.6\cdot 10^{16}$

Lifetime $\tau \sim w^{-1} \sim 10^{-9}$ sec

__Note__: The 2S transition (forbidden by our simplest selection rules) goes by a 2 photon process and has a lifetime $\sim 1/7$ sec!

Presumably this is all the background for things you need to do in HEP