$\boxed{\text{Generating Functions}}$

Encounter many collections of special functions in physics

One way to unify them was already discussed:

Sturm Liouville theory, They are all eigenfunctions

of particular Hermitian 2nd order differential operators

orthogonality, completeness etc

Probably the most useful procedure for deriving

properties of collections of special functions is

the "generating function"

$$g(x,t) = \sum_n \phi_n(x)\, t^n \qquad \text{sometimes } \phi_n(x)\frac{t^n}{n!}$$

expand in powers of t

coefficients give

special functions of interest

EX: Legendre

$$\frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(x)\,t^n$$

$$(1-2xt+t^2)^{-1/2}$$

$$= 1 + \left(-\tfrac{1}{2}\right)(-2xt+t^2)^1 + \tfrac{1}{2}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)(-2xt+t^2)^2 + \ldots$$

$$(1+u)^n = 1 + nu + \tfrac{1}{2}n(n-1)u^2 + \ldots$$

$$= 1 + xt - \tfrac{1}{2}t^2 + \tfrac{3}{8}(-2xt)^2$$

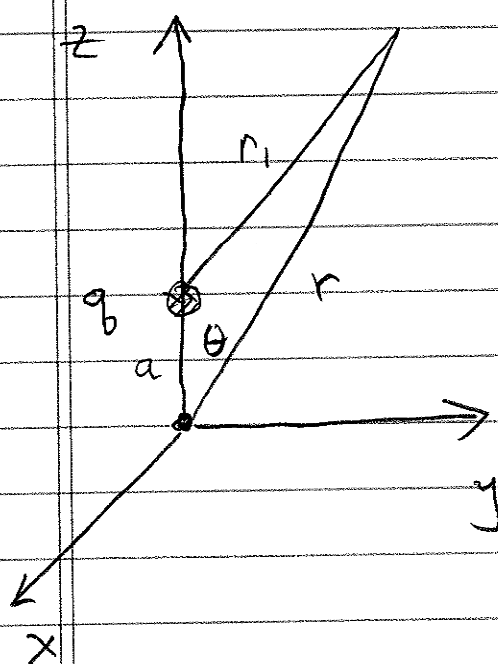
$$= 1 + xt + \left(-\tfrac{1}{2} + \tfrac{3}{2}x^2\right)t^2 + \ldots$$

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2}$$

It is easy to see where this particular generating function comes from



$$V(r) = \frac{q}{4\pi\epsilon_0} \frac{1}{r_1}$$

$$r_1 = \left(r^2 + a^2 - 2ar\cos\theta\right)^{1/2}$$

$$= r\left(1 - 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2}\right)^{1/2}$$

$x \Rightarrow \cos\theta$

$t \Rightarrow a/r$

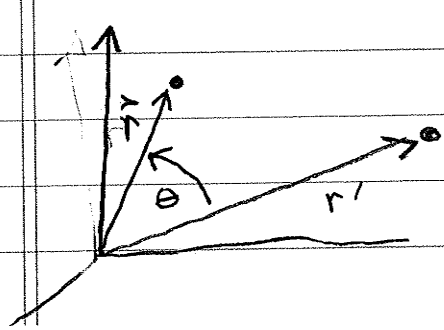
$$V(r) = \frac{q}{4\pi\epsilon_0 r} \frac{1}{\left(1 - 2\cos\theta \frac{a}{r} + \frac{a^2}{r^2}\right)^{1/2}}$$

$$= \frac{q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n$$

This is a special case of the identity

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{n=0}^{\infty} \frac{r_<^n}{r_>^{n+1}} P_n(\cos\theta)$$

$r_<$, $r_>$ are smaller, larger of $|\vec{r}|$ $|\vec{r}\,'|$



Let's show, for example that $P_n(1) = 1$ for all n.

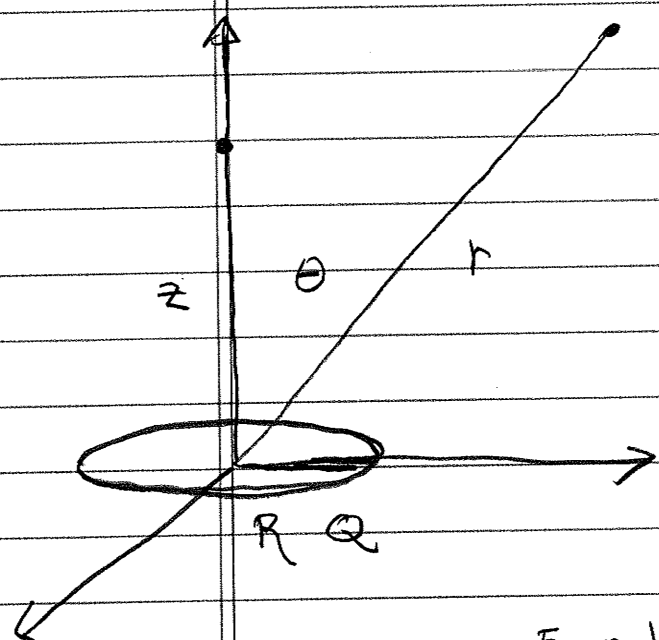
$$\frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

set $x = 1$ $\quad 1-2t+t^2 = (1-t)^2$

$$(1-t)^{-1} = \sum_{n=0}^{\infty} P_n(1) t^n$$

$$1 + t + t^2 + t^3 + \ldots \implies P_n(1) = 1.$$

Ring of charge EM Problem



clearly for $\theta = 0$
(along $\hat{z}$ axis)

$$V = \frac{Q}{4\pi\epsilon_0 (z^2+R^2)^{1/2}}$$

Applying move general form at bottom of page GF-3

For r along $\hat{z}$ axis: $V(z) = \frac{Q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{R^n}{z^{n+1}} P_n(0)$

$\cos \pi/2$

But we must also have

$$V(r) = \frac{Q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{R^n}{r^{n+1}} P_n(\cos\theta)$$

Bessel!

Let's give a few more examples

$$g(x t) = e^{x/2(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

unlike Legendre case $J_n(x)$ are not finite degree polynomials but have all orders. It's a bit of work to get $J_n(x)$ out. (A page or so of algebra)

Let's do something a bit simpler. Show how recursion relation comes from generating function.

$$\partial/\partial t \; g(x t) = \frac{\partial}{\partial t} e^{x/2(t-1/t)}$$

$$= \frac{1}{2} x \left(1 + 1/t^2\right) e^{x/2(t-1/t)}$$

$$\sum_n J_n(x) n t^{n-1} = \frac{1}{2} x \sum \left(1 + 1/t^2\right) J_n(x) t^n$$

$$\sum_n J_{n+1}(n+1) t^n = \frac{1}{2} x \sum J_n(x) t^n + J_n(x) t^{n-2}$$

$$= \frac{1}{2} x \sum J_n(x) t^n + J_{n+2}(x) t^n$$

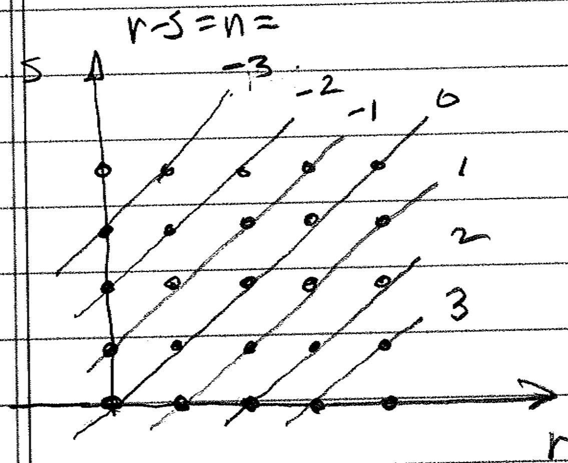
$$(n+1) J_{n+1}(x) = \frac{1}{2} x \left(J_n(x) + J_{n+2}(x)\right)$$

$$n J_n(x) = \frac{1}{2} x \left(J_{n-1}(x) + J_{n+1}(x)\right)$$

Getting power series for $J_n(x)$

$$g(x,t) = e^{x/2(t-1/t)} = e^{xt/2}\, e^{-x/2t}$$

$$= \sum_{r=0}^{\infty} \left(\frac{x}{2}\right)^r \frac{t^r}{r!} \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^s \frac{t^{-s}}{s!} (-1)^s$$



t^{r-s}

$r = n+s$

$r-s=n$

Can get any $n=-\infty, \dots +\infty$

if $n \geq 0$ s starts at $\emptyset$

$$\sum_{n<0} (\;) + \sum_{n\geq 0} \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{n+s} \frac{t^{s+n}}{(s+n)!} \left(\frac{x}{2}\right)^s (-1)^s \frac{t^{-s}}{s!}$$

consider later

$$\sum_{n\geq 0} t^n \underbrace{\sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{n+2s} (-1)^s \, / \, s!(n+s)!}_{J_n(x)}$$

Compare to $\sum_{s=0}^{\infty} x^{2s} (-1)^s / (2s)!$

for $\cos x$

GF-7

$$J_n(x) = \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{n+2s} (-1)^s \,1/s!\,(n+s)!$$

$$J_0(x) = \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^{2s} (-1)^s \,1/(s!)^2$$

$$= \underset{\substack{\uparrow \\ s=0}}{1} + \underset{\substack{\uparrow \\ s=1}}{\left(\frac{x}{2}\right)^2 (-1)} + \underset{\substack{\uparrow \\ s=2}}{\left(\frac{x}{2}\right)^4 1/4}$$

$$= 1 - x^2/2 + x^4/64$$

$$\cos x = 1 - x^2/2 + x^4/24 - x^6/720$$

Let's prove an orthogonality identity

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx$$

A final example is Hermite

$$e^{2tx - t^2} = \sum_{n=0}^{\infty} H_n(x)\, t^n / n!$$

$$\int_{-\infty}^{\infty} e^{2tx-t^2} e^{2ux-u^2} e^{-x^2} dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2 + 2(t+u)x - t^2 - u^2} dx$$

$$= \int_{-\infty}^{\infty} e^{-[x-(t+u)]^2 + (t+u)^2 - t^2 - u^2} dx$$

$$= \sqrt{\pi}\, e^{2tu} = \sqrt{\pi} \sum_{\ell=0}^{\infty} (2tu)^\ell / \ell!$$

But this must also equal

$$\sum_{n\,m} \int_{-\infty}^{\infty} dx\, e^{-x^2} H_n(x) \frac{t^n}{n!} H_m(x) \frac{u^m}{m!} = \sqrt{\pi} \sum_{\ell=0}^{\infty} \frac{(2tu)^\ell}{\ell!}$$

Since t, u only appear with same power on rhs

we must have $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx \sim \delta_{nm}$

We can even get value for $n=m$

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} H_n(x) H_m(x) = (n!)^2 \sqrt{\pi}\, 2^n / n!$$

$$= \sqrt{\pi}\, 2^n n!$$

Perturbation Theory in Field Theory (Feynman Diagrams)

Fundamental identity is "Feynman Disentangling Thm"

$$\hat{H} = \hat{H}_0 + \hat{V}$$

$$e^{-i\hat{H}t} = e^{-i\hat{H}_0 t}\, T\exp -i\int_0^t \hat{V}(t')\,dt'$$

"time ordered exponential" ?!?

$$\hat{V}(t') = e^{i\hat{H}_0 t}\,\hat{V}\,e^{-i\hat{H}_0 t}$$

$$e^{-i\hat{H}t} = e^{-i\hat{H}_0 t}\,\hat{U}(t)$$

time evolution due to $\hat{H}_0$

"small" ie close to I operator

$$\hat{U}(t) = e^{i\hat{H}_0 t}\,e^{-i\hat{H}t}$$

$$\frac{d\hat{U}}{dt} = e^{i\hat{H}_0 t}\,\hat{H}_0\,e^{-i\hat{H}t} - e^{i\hat{H}_0 t}\,\hat{H}\,e^{-i\hat{H}t}$$

$$= e^{i\hat{H}_0 t}\,\hat{V}\,e^{-i\hat{H}t}$$

$$= -i e^{i\hat{H}_0 t}\,\hat{V}\,e^{-i\hat{H}_0 t}\,e^{i\hat{H}_0 t}\,e^{-i\hat{H}t}$$

$$= -i\hat{V}(t)\,\hat{U}(t)$$

PTFT-2

Convert this differential eqn into integral eqn

$$\hat{U}(t) = \mathbb{I} - i\int_0^t \hat{V}(t_1)\hat{U}(t_1)\,dt_1$$

$\hat{U}(t=0) = \mathbb{I}$ $\quad$ (see $e^{-i\hat{H}t} = e^{-i\hat{H}_0 t}\,U(t)$

can also do algebra with

$\hat{U}(t_0) = \mathbb{I}$)

Solve iteratively

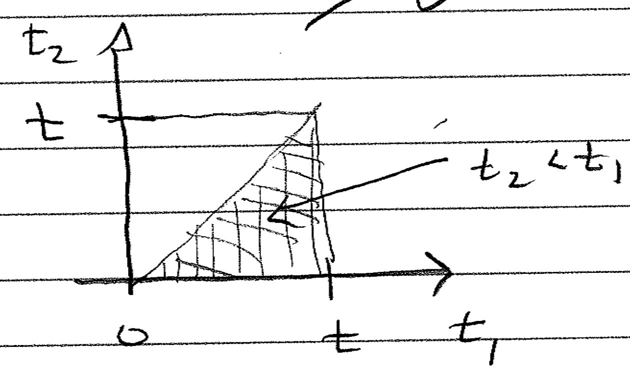
$$\hat{U}_0(t) = \mathbb{I}$$

plug in $\hat{U}(t_1) = \mathbb{I}$

$$\hat{U}_1(t) = \mathbb{I} - i\int_0^t \hat{V}(t_1)\,dt_1$$

$$U_2(t) = \mathbb{I} - i\int_0^t dt_1\, \hat{V}(t_1)\left\{\mathbb{I} - i\int_0^{t_1}\hat{V}(t_2)\,dt_2\right\}$$

$$= \mathbb{I} - i\int_0^t dt_1\, \hat{V}(t_1) + (-i)^2\int_0^t dt_1\int_0^{t_1} dt_2\, \hat{V}(t_1)\hat{V}(t_2)$$



By symmetry last term is

$$(-i)^2 \ \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 \quad \hat{V}(t_1)\,\hat{V}(t_2)$$

$\boxed{\text{BUT}}$ in original integral $\underset{\uparrow \atop \text{later}}{\hat{V}(t_1)}\ \underset{\uparrow \atop \text{earlier}}{\hat{V}(t_2)}$

So need really

$$(-i)^2 \quad \frac{1}{2}\int_0^t dt_1 \int_0^t dt_2 \ \underset{\uparrow}{T}\ \hat{V}(t_1)\,\hat{V}(t_2)$$

time ordering operator

→ latest time to left

$$\text{So} \quad e^{-i\hat{H}t} = e^{-i\hat{H}_0 t}\ T\exp -i\int_0^t \hat{V}(t_1)\,dt_1$$

* This calculation is a preview of Physics 230, 240C

also hints at importance/usefulness of converting

differential eqn to integral eqn

$$\frac{d\hat{u}}{dt} = -i\hat{V}(t)\,\hat{u}(t) \qquad \hat{u}(t) = I - i\int_0^t \hat{V}(t_1)\,\hat{u}(t_1)\,dt_1$$