We saw a modern problem in CM physics was to figure out ground state of Heisenberg model on various lattices. We would look for AF by trying to see if knowledge of spin direction "persists" far away. If so $\rightarrow$ magnetic order

Nowadays a hot topic is phases which do not have such a clear criterion for order. It is proposed that the "Entanglement Entropy" should discern transitions to and from such exotic phases.

Prof Cheng introduced density matrix (for pure state)

$$\hat{\rho} = |u\rangle\langle u|.$$

Density matrix for ground state of $N=2$ site Heisenberg

$$\rho = |\psi_0\rangle\langle\psi_0|$$

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

We will push this a little further.

"Entanglement" — how is quantum state in one region of lattice related to that in another region?

Answer. Compute "Entanglement Entropy"

There are different definitions, we will use one due to Rényi:

(1) Start with full density matrix $\hat{\rho}$

(2) Partition system into 2 regions A/B

(3) Compute "reduced density matrix"

$$\hat{\rho}_A \equiv \mathrm{Tr}_B\, \hat{\rho}$$

(4) $S \equiv -\mathrm{Tr}_A\, \hat{\rho}_A^{\,2}$

Emphasize: Hot topic in CMP but much more!

Thermodynamics of black holes exact same quantity being examined

and also in HE field theory generally!

Another notation basis vectors $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

then
$$\hat{\rho} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Q. What did you learn about $\hat{\rho}$?

A. Well $tr\,\hat{\rho} = 1$. Is that true here?

Also $tr\,\hat{\rho}^2 \le 1$ True here?

$$\hat{\rho}^2 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \hat{\rho}\ !$$

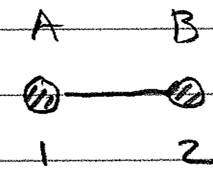
EE-4

What do steps (3) and (4) really mean?

An example always helps.

$N=2$ site Heisenberg

$$\hat{\rho} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$




matrix elements $\langle +-|\hat{\rho}|+-\rangle = 1$

$\langle +-|\hat{\rho}|-+\rangle = -1$

etc

$\hat{\rho}_A = 2\times 2$ matrix

spin 1

trace over $B$ = spin 2

$$\langle +|\hat{\rho}_A|+\rangle = \langle ++|\hat{\rho}|++\rangle + \langle +-|\hat{\rho}|+-\rangle$$

A (spin 1) indices

$$= 0 + 1/2 \qquad \hat{\rho}_A = \begin{pmatrix} 1/2 & 0 \\ \cdot & \cdot \end{pmatrix}$$

$$\langle +|\hat{\rho}_A|-\rangle = \langle ++|\hat{\rho}_A|-+\rangle + \langle +-|\hat{\rho}_A|--\rangle = 0+0$$

EE-5

HW is to finish up calculation and get entanglement entropy. (Answer $\hat{\rho}_A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$)

$$S_A = -\ln \operatorname{tr} \rho_A^2$$

$$= -\ln(1/4 + 1/4) = -\ln 1/2 = \ln 2 .$$

fully entangled ?

## The Density Matrix J. von Neumann 1927

Consider a system in a given state described by $|u\rangle$.
It can also be described by $|v\rangle = e^{i\theta}|u\rangle$
The density operator of this system is defined by

$$\rho = |u\rangle\langle u| = e^{i\theta}|u\rangle e^{-i\theta}\langle u| = |v\rangle\langle v|$$

It can also be used to describe this state
For a given basis $|\phi_i\rangle$. it can be expressed as the density matrix with elements

$$\langle\phi_i|u\rangle\langle u|\phi_j\rangle$$

Some properties:

(a) $\mathrm{tr}\,\rho = \sum_i \langle\phi_i|u\rangle\langle u|\phi_i\rangle = \sum_i \langle u|\phi_i\rangle\langle\phi_i|u\rangle = \langle u|u\rangle$

(b) The expectation value of any observable $A$ is

$$\langle A\rangle = \frac{\langle u|A|u\rangle}{\langle u|u\rangle} = \mathrm{tr}(A\rho)/\mathrm{tr}(\rho)$$

$$\mathrm{tr}(\rho A) = \sum_i \langle\phi_i|u\rangle\langle u|A|\phi_i\rangle = \sum_i \langle u|A|\phi_i\rangle\langle\phi_i|u\rangle = \langle u|A|u\rangle$$

(c) If $\langle u|u\rangle = 1$

$\rho^2 = \rho$ for a pure state.

$$\rho^2 = |u\rangle\langle u|u\rangle\langle u| = |u\rangle\langle u| = \rho$$

The density operator is useful because it is readily generalized to describe mixed ensembles, e.g. a beam of uncorrelated particles, partially polarized light. It is useful in quantum statistical mechanics.

Pure ensemble : a collection of physical systems with every member is characterized by the same state vector $|u\rangle$

Mixed ensemble : a fraction of the members with relative population $w_1$ in $|u_1\rangle$ , some other fraction with relative population $w_2$ in $|u_2\rangle$ and so on

$$\sum_a w_a = 1$$

$|u_1\rangle \; |u_2\rangle \cdots$ need not be orthogonal

The density operator in this case is defined by

$$\rho \equiv \sum_a w_a |u_a\rangle\langle u_a|$$

and the density matrix :

$$\langle\phi_i|\rho|\phi_j\rangle = \sum_a w_a \langle\phi_i|u_a\rangle\langle u_a|\phi_j\rangle$$

Suppose we make a measurement on a mixed ensemble of some observable $A$. The average measured value of $A$ when a large number of measurements are carried out is given by the ensemble average of $A$

$$\bar{A} = \sum_a w_a \langle u_a|A|u_a\rangle = \sum_{aij} w_a \langle u_a|\phi_i\rangle\langle\phi_i|A|\phi_j\rangle\langle\phi_j|u_a\rangle$$

$$= \sum_{ij}\left(\sum_a w_a \langle\phi_j|u_a\rangle\langle u_a|\phi_i\rangle\right)\langle\phi_i|A|\phi_j\rangle$$

$$= \sum_{ij}\langle\phi_j|\rho|\phi_i\rangle\langle\phi_i|A|\phi_j\rangle$$

$$= \mathrm{tr}(\rho A)$$

$$tr(\rho) = \sum_a \sum_i w_a \langle \phi_i | u_a \rangle \langle u_a | \phi_i \rangle$$
$$= \sum_a w_a \langle u_a | u_a \rangle$$
$$= 1 \quad \text{if } \langle u_a | u_a \rangle \text{ are normalized to } 1$$

$tr(\rho^2) \leq 1$ equality holds for a pure ensemble.

$$tr(\rho^2) = \sum_{ab}\sum_{ij} w_a \langle \phi_i | u_a \rangle \langle u_a | \phi_j \rangle w_b \langle \phi_j | u_b \rangle \langle u_b | \phi_i \rangle$$
$$= \sum_{ab} w_a w_b \langle u_a | u_b \rangle \langle u_b | u_a \rangle$$
$$= \sum_{ab} w_a w_b |\langle u_a | u_b \rangle|^2$$
$$\leq \sum_{ab} w_a w_b = 1$$

equality holds iff $|\langle u_a | u_b \rangle| = 1$

Examples 1) A completely polarized beam with $S_z +$

$$\rho = |+\rangle\langle+| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

2) A completely polarized beam with $S_x +$

$$\rho = |S_x +\rangle\langle S_x +| = \left[\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\right]\left[\frac{1}{\sqrt{2}}(\langle+| + \langle-|)\right]$$
$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

3) An unpolarized beam. This can be regarded as an incoherent mixture of a spin-up ensemble and a spin-down ensemble with equal weights (50%)

$$\rho = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-|$$
$$= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$