Crucial question in teaching physics 215 (or perhaps <u>any</u> course!) is balance between "legacy" material and modern (research) topics. In QM such "legacy" problems are, for example, full sol'n of H atom, whereas current subjects are topological insulators, graphene, iron-pnictide superconductors (in CM physics) or the Higgs boson, string theory, (in HE).

A key issue is the extent to which legacy knowledge is required to understand new areas.

⇛ My sol'n will be to cut back on some of tedious derivations of legacy QM eg detailed derivation of Laguerre polynomials of H atom. Possible danger: you won't know how to do such derivations when you encounter their analogs in today's research.

R2

Review of 215A

Prof Cheng covered material pretty much as I would have.

Still, let's summarize key points briefly.

(1) State of QM system described by vector $|\psi\rangle$

in "Hilbert space" $\leftarrow$ ∞ dim vector space

(2) To each physical observable is associated a

Hermitian operator. Possible results of measurements are eigenvalues.

Q: What is definition of Hermitian?

A: In Hilbert space we have inner product $\langle\phi|\psi\rangle$

Hermitian: $\langle\phi|\hat{H}|\psi\rangle = \langle\psi|\hat{H}|\phi\rangle^*$

Q: What is significance of Hermitian?

A: Real eigenvalues; complete set eigenvectors

also these are orthogonal

(3) $[\hat{x}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$

(4) A $\quad |\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$ where $\hat{H}$ is Hamiltonian

B $\quad i\hbar\, d\hat{A}/dt = [\hat{A}, \hat{H}]$

A, B are two equivalent ways of doing QM

Q: A, B called what?

A: Shruedinger & Heisenberg pictures

Q: other ways of doing QM?

A: "Interaction picture", path integrals

Q: Can you prove Sch / Heis equivalence?

A: Soln to (B) is

$$\hat{A}(t) = e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar}$$

Schr: $\langle \hat{A}(t) \rangle = \langle \psi(t) | \hat{A} | \psi(t) \rangle$

$$= \langle \psi(0) | e^{+i\hat{H}t/\hbar} A e^{-i\hat{H}t/\hbar} | \psi(0) \rangle$$

Schroedinger

$$= \langle \psi(0) | \hat{A}(t) | \psi(0) \rangle$$

Heisenberg

In undergrad QM "Schroedinger Eqn"

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x,t) + V(x)\psi(x,t) = i\hbar \frac{\partial}{\partial t} \psi(x,t) \quad *$$

Q: What is connection between (1)–(4) and *

A: Schr. Eqn is obtained by picking a particular basis for Hilbert space $\hat{x}|x\rangle = x|x\rangle$ complete set of eigenvectors

position op

What is $\psi(x)$? It is $\psi(x) = \langle x|\psi\rangle$

"components of $|\psi\rangle$ in eigenstates of position operator"

N.B. Almost all calculations in a vector space end up getting done by picking a basis!

Q: What are components of eigenstate $|p\rangle$ of momentum operator $\hat{p}$ in the basis of eigenstates of $\hat{X}$

A: $\langle x|p\rangle = e^{-ipx/\hbar}/\sqrt{2\pi\hbar}$

Q: Given $|\psi\rangle$ and $\psi(x) = \langle x|\psi\rangle$ what are components of $\hat{p}|\psi\rangle$?

A: $\langle x|\hat{p}|\psi\rangle = -\frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x)$

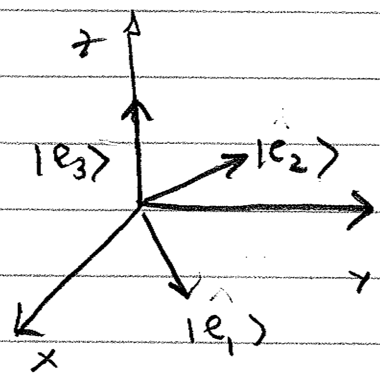
change of basis is usual linear algebra thing

$$\tilde{\psi}(p) = \langle p|\psi\rangle = \int \langle p|x\rangle\langle x|\psi\rangle\, dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}\psi(x)\, dx \quad \text{Fourier transform}$$

$$I = \int dx\, |x\rangle\langle x|$$

Q: Any mystery about $I = \int dx\, |x\rangle\langle x|$?



basis $|\hat{e}_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$

$|e_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}$

$|e_3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}\frac{1}{\sqrt{2}}(1\ 1\ 0) + \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}\frac{1}{\sqrt{2}}(-1\ 1\ 0) + \begin{pmatrix}0\\0\\1\end{pmatrix}(0\ 0\ 1)$$

$$= \frac{1}{2}\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1&-1&0\\-1&1&0\\0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} = I \ \checkmark\checkmark$$

R6

To actually solve a QM problem:

① Get eigenstates of $\hat{H}$ $\quad \hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$

② Given $|\psi(t=0)\rangle$ compute $c_n = \langle\phi_n|\psi(t=0)\rangle$

③ $|\psi(t)\rangle = \sum_n c_n e^{-iE_nt/\hbar} |\phi_n\rangle$

Can work with any basis but eigenstates of $\hat{H}$ are special because of time evolution

Q: where does ③ come from?

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle$$

$$= \sum_n \underbrace{e^{-i\hat{H}t/\hbar} |\phi_n\rangle}_{e^{-iE_nt/\hbar} |\phi_n\rangle} \underbrace{\langle\phi_n|\psi(0)\rangle}_{c_n}$$

Commutation Relations $[\hat{x},\hat{p}] = i\hbar$ put conditions on measurements.

Q: what condition?

A: $\Delta x \Delta p \geq \hbar/2$

Q: what is Δx?

A: $\sqrt{\langle\hat{x}^2\rangle - \langle x\rangle^2} = \sqrt{\langle\psi|\hat{x}^2|\psi\rangle - \langle\psi|x|\psi\rangle^2}$

"Levi-Civita" symbol

Approaching end of 215A

$$[\hat{J}_i, \hat{J}_j] = i\hbar \epsilon_{ijk} \hat{J}_k \qquad [\hat{J}_i, \hat{J}^2] = 0$$

"Angular momentum operators"

Can prove these for orbital AM $\hat{L} = \hat{r} \times \hat{p}$

from $[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$

Just as $[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$ carried implication

$\Delta x \Delta p \geq \hbar/2$, AM comm reln's:

$$\hat{J}^2 |j m\rangle = \hbar^2 j(j+1) |j m\rangle$$

$$\hat{J}_z |j m\rangle = \hbar m |j m\rangle$$

where Q: ? A: j = integer or half integer

$m = -j, -j+1, \ldots +j$

Amazing?! Comm reln's $\Rightarrow$ eigenspectrum

R7A

← Sum on j understood

$$[\hat{J}_i, \hat{J}^2] = [\hat{J}_i, \hat{J}_j \hat{J}_j]$$

$$= [\hat{J}_i, \hat{J}_j] \hat{J}_j + \hat{J}_j [\hat{J}_i, \hat{J}_j]$$

$$= i\hbar \epsilon_{ijk} \hat{J}_k \hat{J}_j + \hat{J}_j \, i\hbar \epsilon_{ijk} \hat{J}_k$$

$$= i\hbar \epsilon_{ijk} (\hat{J}_k \hat{J}_j + \hat{J}_j \hat{J}_k)$$

changes sign under jk interchange

does not change sign

$\Rightarrow \phi$

proof? Rename dummies $j \leftrightarrow k$

$$= i\hbar \epsilon_{ikj} (\hat{J}_j \hat{J}_k + \hat{J}_k \hat{J}_j)$$

$$= -i\hbar \epsilon_{ijk} (\hat{J}_k \hat{J}_j + \hat{J}_j \hat{J}_k)$$

R7B

$$[\hat{L}_a, \hat{L}_b] = [\epsilon_{aij} \hat{x}_i \hat{p}_j, \epsilon_{bk\ell} \hat{x}_k \hat{p}_\ell]$$

$$= \epsilon_{aij} \epsilon_{bk\ell} [\hat{x}_i \hat{p}_j, \hat{x}_k \hat{p}_\ell]$$

$$[\hat{A}\hat{B}, \hat{C}\hat{D}] = ABCD - CDAB$$ "drop hats"

$$= ABCD - ACBD + ACBD - CDAB$$

$$= A[B,C]D + ACBD - ACDB + ACDB - CDAB$$

$$= A[B,C]D + AC[B,D] + ACDB - CADB + CADB - CDAB$$

$$= A[B,C]D + AC[B,D] + [A,C]DB + C[A\,D]B$$

So $[\hat{x}_i \hat{p}_j, \hat{x}_k \hat{p}_\ell]$

$$= i\hbar(-x_i \delta_{jk} p_\ell + x_i x_k \phi + \phi p_\ell p_j + x_k \delta_{i\ell} p_j)$$

$$[\hat{L}_a, \hat{L}_b] = \epsilon_{aij} \epsilon_{bk\ell} i\hbar(-\delta_{jk} x_i p_\ell + \delta_{i\ell} x_k p_j)$$

$$= i\hbar(-\epsilon_{aij}\epsilon_{bj\ell} x_i p_\ell + \epsilon_{aij}\epsilon_{bki} x_k p_j)$$

$$+\epsilon_{aij}\epsilon_{b\ell j} \qquad -\epsilon_{aji}\epsilon_{bki}$$

Levi Civita Identity

$$+(\delta_{ab}\delta_{i\ell} - \delta_{a\ell}\delta_{ib})(x_i p_\ell)$$

$$\epsilon_{ijk}\epsilon_{rsk} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}$$

$$-(\delta_{ab}\delta_{jk} - \delta_{ak}\delta_{bj}) x_k p_j$$

R7C

$$[\hat{L}_a, \hat{L}_b] = i\hbar\left(\cancel{\delta_{ab} x_\ell p_\ell} - x_b p_a - \cancel{\delta_{ab} x_j p_j} + x_a p_b\right)$$

cancel

$$= i\hbar\left(x_a p_b - x_b p_a\right)$$

Meanwhile $\quad i\hbar\, \epsilon_{abk} \hat{L}_k = i\hbar\, \epsilon_{abk} \epsilon_{ijk} x_i p_j$

$$= i\hbar\left(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}\right) x_i p_j$$

$$= i\hbar\left(x_a p_b - x_b p_a\right)$$

R8

NOTE TO ME:

215A Ang Mom background

1) AM and rotations in 3D space $\rightarrow$ comm relns
2) Eigenspectrum from comm relns
3) spin 1/2
4) precession; time evolution in $\mathcal{H} = -\gamma \vec{J} \cdot \vec{B}$
5) n-interferometry
6) NMR
7) spin 1
8) orbital AM / spherical Harmonics
9) Addition of AM
    * 2 spin 1/2