Born Approximation

We are attempting to solve

$$\left\{ -\frac{\hbar^2}{2m}\nabla^2 + V(r) \right\} \phi(r) = E\,\phi(r)$$

Writing $E = \hbar^2 k^2 / 2m$ and $u(r) = \frac{2mV(r)}{\hbar^2}$

$$(\nabla^2 + k^2)\,\phi(r) = u(r)\,\phi(r)$$

Suppose we knew

$$(\nabla^2 + k^2)\,G(r) = \delta(r) \qquad *$$

Q: What is G called?

Then we can write

$$\phi(r) = \phi_0(r) + \int d^3r'\, G(r-r')\,\phi(r')\,u(r')$$

$$(\nabla^2 + k^2)\,\phi_0(r) = 0$$

Q: What $\phi_0(r)$ might this be?
A: e^{ikz}

Q: Does anyone know G for this Eqn $*$?

$$G(r) = -\frac{1}{4\pi}\,\frac{e^{\pm ikr}}{r}$$

B-1A

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad r = (x^2+y^2+z^2)^{1/2}$$

$$\frac{\partial}{\partial x}\frac{e^{ikr}}{r} = \frac{e^{ikr}}{r^3}\left(\frac{-1}{2}\right)(2x) + \frac{1}{r}e^{ikr}(ik)\frac{1}{2}\frac{1}{r}2x$$

$$= \frac{xe^{ikr}}{r^3}\{-1+ikr\}$$

$$\frac{\partial^2}{\partial x^2}\frac{e^{ikr}}{r} = \frac{e^{ikr}}{r^3}\{-1+ikr\} + x\frac{e^{ikr}}{r^3}ik\frac{1}{\cancel{2}r}\cancel{2}x\{-1+ikr\}$$

$$+ \frac{xe^{ikr}}{r^5}\left(-\frac{3}{2}\right)2x\{-1+ikr\} + \frac{xe^{ikr}}{r^3}\left\{ik\frac{1}{\cancel{2}r}\cancel{2}x\right\}$$

$$= \frac{e^{ikr}}{r^5}\{-r^2+ikr^3+ikx^2r - k^2x^2r^2$$

$$+3x^2-3x^2ikr+x^2rik\}$$

$$= \frac{e^{ikr}}{r^5}\{-k^2x^2r^2 - r^2+3x^2+ikr(r^2-3x^2)\}$$

Adding the $\frac{\partial^2}{\partial y^2}$ and $\frac{\partial^2}{\partial z^2}$ terms

$$\nabla^2\left(\frac{e^{ikr}}{r}\right) = \frac{e^{ikr}}{r^5}\{-k^2(x^2+y^2+z^2)r^2 \quad \leftarrow -k^2r^4$$

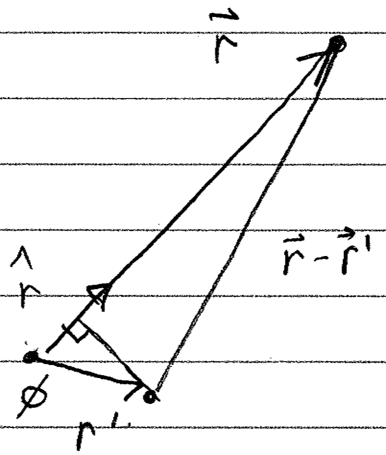
$$-3r^2+3(x^2+y^2+z^2) \quad \leftarrow \phi$$

$$+ikr(3r^2-3(x^2+y^2+z^2)) \quad \leftarrow \phi$$

$$= -k^2\frac{e^{ikr}}{r} \Rightarrow (\nabla^2+k^2)\frac{e^{ikr}}{r} = \phi$$

$\int d^3r' \, G(r-r') \, \phi(r') \, u(r')$

potential has
some finite range



$$|\vec{r}-\vec{r}'| \approx |r| - \hat{r}\cdot\vec{r}'$$

$$G(r-r') = -\frac{1}{4\pi} \frac{e^{ikr}}{r} e^{-ik\hat{r}\cdot\vec{r}'}$$

$$\phi(r) = \phi_0(r) + \int d^3r' \, \frac{e^{ikr}}{r} \, \frac{-1}{4\pi} e^{ik\hat{r}\cdot\vec{r}'} u(r') \, \phi(r')$$

Q: eg e^{ikz}

$$= e^{ikz} + \frac{e^{ikr}}{r}\left[\frac{-1}{4\pi}\int d^3r' e^{-ik\hat{r}\cdot\vec{r}'} u(r')\phi(r')\right]$$

$f_k(\theta, \phi)$

This is a more rigorous derivation of the solution we originally assumed.

In general $\phi_0(r) = e^{i\vec{k}_i\cdot\vec{r}}$

$$f_k(\theta,\phi) = -\frac{1}{4\pi}\int d^3r'\, e^{-ik\hat{r}\cdot\vec{r}\,'}\, u(r')\,\phi(r')$$

Born approximation $\qquad\qquad \phi(r') \downarrow e^{i\vec{k}_i\cdot\vec{r}\,'}$

$\vec{k}_f = k\hat{r}$ because we are looking for scattered particles at $\hat{r}$

$$\boxed{f_k(\theta,\phi) = -\frac{1}{4\pi}\int d^3r'\, e^{-i(\vec{k}_f-\vec{k}_i)\cdot r'}\, u(r')}$$

One of most useful Eqns in physics!

Very reasonable physically

Probability amplitude to scatter $\vec{k}_f - \vec{k}_i = \vec{\Delta k}$

is given by Fourier component of $u(r')$

of that $\vec{\Delta k}$ value!

Eg underlies all neutron scattering

determination of crystal structure.

$$u(r') = \sum_{R_n} u(r'-R_n)$$

← nuclear positions