Physics 215B- Quantum Mechanics, Winter 2014

Problem Set 5, due Thursday March 6

[1.] *Qualifier Problem!* A system with unperturbed eigenstates $\phi_n(x)$ and energies E_n is subject to a perturbation

$$\hat{V} = \frac{\hat{A}}{\sqrt{\pi}\tau} e^{-t^2/\tau^2}$$

where $\hat{A}$ is a time-independent operator.

a) If at $t = -\infty$ the system is in its ground state ϕ_0, show that, to first order, the *probability amplitude* that at $t = +\infty$ the system will be in its m-th excited state ($m \neq 0$) is:

$$c_m(+\infty) = -i \frac{\langle m \,|\, \hat{A} \,|\, 0 \rangle}{\hbar} \exp\left(-\frac{\tau^2}{4\hbar^2}(E_0 - E_m)^2 \right)$$

b) Next consider the limit of an impulsive perturbation $\tau = 0$ and compute the *probability* P that the system makes any transition out of the ground state.

[2.] *Qualifier Problem!* A particle of charge e and mass m is bound in a three dimensional harmonic oscillator potential $V = \frac{1}{2}m\omega^2 r^2$. We wish to calculate the lifetime of the first excited state.

a) Show that the condition for the electric dipole approximation to be valid is $\hbar\omega << mc^2$.

b) Calculate the lifetime of the first excited state in this limit.

c) The first excited state is degenerate. What effect (if any) does this have on the calculation in part b above?

[3.] *Qualifier Problem!* A one microampere beam of 100 keV electrons is incident on a 1 cm thick target of hydrogen of density 10^{-3} g/cm^3. Approximate the interaction potential between an electron and a hydrogen atom by the screened Coulomb interaction

$$V(r) = -\frac{e^2}{r} e^{-r/a_0}$$

where a_0 is the Bohr radius. Using the first Born approximation, calculate the number of electrons per second scattered elastically into a detector at an angle of 10 mrad to the incident beam and which subtends a solid angle of 10^{-5} steradians.

[4.] *Qualifier Problem!* Consider the scattering of a particle of mass m off a spherical well of radius R and depth V_0.

a) Find the total cross section for the scattering of very low energy particles off the well.

b) Find the total cross section for very high energies.

You may leave your answers in terms of precisely defined integrals of known functions.

P215B Winter 2014
Problem Set 5

$\boxed{1}$ We have the Schroedinger eqn

{ NOTE: MUCH OF THIS IS IN CLASS LECTURE MATERIAL }

$$(\hat{H}_0 + \hat{V})|\psi\rangle = i\hbar \frac{\partial}{\partial t}|\psi\rangle$$

where $\hat{H}_0 |\phi_n\rangle = E_n|\phi_n\rangle$ are assumed known

we expand $|\psi\rangle = \sum_n a_n(t) e^{-iE_nt/\hbar} |\phi_n\rangle$

and plug into Sch. Eqn

$$\sum_n a_n(t) e^{-iE_nt/\hbar} E_n|\phi_n\rangle + \sum_n a_n(t) e^{-iE_nt/\hbar}\hat{V}|\phi_n\rangle$$

$$= \sum_n a_n(t) E_n e^{-iE_nt}|\phi_n\rangle + \sum_n i\hbar \dot{a}_n(t) e^{-iE_nt/\hbar}|\phi_n\rangle$$

The first terms on lhs and rhs cancel.

We can take inner product $\langle\phi_m|$ and use $\langle\phi_m|\phi_n\rangle = \delta_{nm}$

$$\sum_n a_n(t) e^{-iE_nt/\hbar}\langle\phi_m|\hat{V}|\phi_n\rangle = i\hbar \dot{a}_m(t) e^{-iE_mt/\hbar}$$

To zeroth order in $\hat{V}$

$$0 = \dot{a}_m^{(0)}(t)$$

$$a_m^{(0)}(t) = \text{const} \leftarrow$$

We are given that $a_m^{(0)}(t=-\infty) = \begin{cases} 1 & m=0 \\ 0 & \text{otherwise} \end{cases}$

1-2

To first order in $\hat{V}$

$$\mathring{a}_m^{(1)}(t) = \frac{1}{i\hbar} e^{+iE_m t/\hbar} \sum_n a_n^{(0)}(t) e^{-iE_n t/\hbar} \langle \phi_m | \hat{V} | \phi_n \rangle$$

$$= \frac{1}{i\hbar} e^{i(E_m - E_0)t/\hbar} \langle \phi_m | \hat{V} | \phi_0 \rangle$$

because of initial conditions.

Integrating yields

$$a_m^{(1)}(\infty) = \int_{-\infty}^{\infty} \frac{1}{i\hbar} e^{i(E_m - E_0)t/\hbar} \langle \phi_m | \hat{V} | \phi_0 \rangle$$

Using given time dependence

$$= \int_{-\infty}^{\infty} \frac{1}{i\hbar} e^{i(E_m - E_0)t/\hbar} \frac{1}{\sqrt{\pi}\,\tau} e^{-t^2/\tau^2} \langle \phi_m | \hat{A} | \phi_0 \rangle$$

$$-t^2/\tau^2 + i(E_m - E_0)t/\hbar$$

$$= -\frac{1}{\tau^2}\left[t - \frac{i(E_m - E_0)}{2\hbar}\tau^2 \right]^2 - \frac{(E_m - E_0)^2 \tau^2}{4\hbar^2}$$

Doing the Gaussian integral yields $\sqrt{\pi}\,\tau$ so

$$a_m^{(1)}(\infty) = \frac{1}{i\hbar} \langle \phi_m | \hat{A} | \phi_0 \rangle \, e^{-\tau^2 (E_m - E_0)^2 / 4\hbar^2}$$

as desired.

An impulsive perturbation is obtained by letting $\tau \to 0$

Since $\frac{1}{\sqrt{\pi}\,\tau} e^{-t^2/\tau^2} \to \delta(t)$ as $\tau \to 0$

We get $a_m^{(1)}(\infty) = \frac{1}{i\hbar} \langle \phi_m | \hat{A} | \phi_0 \rangle$

The probability of a transition to any $\langle \phi_m |$ is

$$P_\infty = \sum_{m \neq 0} |a_m^{(1)}(\infty)|^2$$

$$= \sum_{m \neq 0} \frac{1}{\hbar^2} |\langle \phi_m | \hat{A} | \phi_0 \rangle|^2$$

We can rewrite this in a cute way

$$\sum_{m \neq 0} |\langle \phi_m | \hat{A} | \phi_0 \rangle|^2 = \underbrace{\sum_{\text{all } m} |\langle \phi_m | \hat{A} | \phi_0 \rangle|^2}_{} - |\langle \phi_0 | \hat{A} | \phi_0 \rangle|^2$$

$$\sum_{\text{all } m} \langle \phi_0 | \hat{A} | \phi_m \rangle \langle \phi_m | \hat{A} | \phi_0 \rangle$$

$$= \langle \phi_0 | \hat{A}^2 | \phi_0 \rangle$$

$$P_\infty = \frac{1}{\hbar^2} \left\{ \langle \phi_0 | \hat{A}^2 | \phi_0 \rangle - |\langle \phi_0 | \hat{A} | \phi_0 \rangle|^2 \right\} \; !$$

Physical interpretation of fluctuation of $\hat{A}$ in ground state!

We will use some of what we discussed in class

Fermi's golden rule:

$$w_{\ell \to n} = \frac{2\pi}{\hbar} |\langle n | \hat{\mathcal{V}} | \ell \rangle|^2 g(E_n)$$

$$\hat{\mathcal{V}} = -e/_{2mc} (\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p}) = \frac{-e}{mc} \vec{A} \cdot \vec{p}$$

if $\vec{\nabla} \cdot \vec{A} = \phi$

$$g(E) = \frac{k^2 dk d\Omega}{(2\pi)^3 d(\hbar\omega)} = \frac{\omega^2}{(2\pi)^3} \frac{d\Omega}{\hbar c^3}$$

The quantized expression for the vector potential

$$\hat{A}(rt) = \sum_{k\alpha} c \sqrt{\frac{\hbar}{2\omega}} \left\{ \hat{a}_{k\alpha} \vec{\epsilon}_\alpha e^{i(\vec{k} \cdot \vec{r} - \omega t)} + \hat{a}^{+}_{k\alpha} \vec{\epsilon}_\alpha e^{-i(\vec{k} \cdot \vec{r} - \omega t)} \right\}$$

We need matrix elements $\langle k | \vec{p} | e \rangle \cdot \vec{\epsilon}_\alpha$

and use the trick $[\hat{H}_0, r_j] = \frac{\hbar}{im} p_j$ so that the

required object is $\frac{im}{\hbar}(E_n - E_e) \langle n | \vec{r} | e \rangle \cdot \vec{\epsilon}_\alpha$

The photon energy ω_{ph} must match up with $(E_k - E_e)/\hbar$

by energy conservation $r_j = \sqrt{\frac{\hbar}{2m\omega}} (a_j + a_j^{+})$

only connects states $\langle n |$ and $| \ell \rangle$ differing by $E_n - \epsilon_\ell = \pm \hbar \omega_{osc}$

Thus $\omega_{ph} = \omega_{osc.} \equiv \omega$.

2-2.

The electric dipole approximation sets $e^{i(\vec{k}\cdot\vec{r})} \to 1$

by assuming $\vec{k}\cdot\vec{r}$ is small. In this problem

$$\langle n|\vec{r}|e\rangle \sim \sqrt{\hbar/2m\omega} \quad \text{while} \quad k \sim \omega/c$$

so $$kr \sim \frac{\omega}{c}\sqrt{\frac{\hbar}{2m\omega}} \sim \sqrt{\frac{\hbar\omega}{mc^2}}$$

So ED approximation requires $\hbar\omega \ll mc^2$.

In expression for transition rate we have

$$\sum_\alpha |\langle n|\vec{r}\cdot\hat{\epsilon}_\alpha|e\rangle|^2 \quad \text{but } \hat{\epsilon}_1, \hat{\epsilon}_2, \hat{k} \text{ form a basis}$$

so $$(\vec{r}\cdot\hat{\epsilon}_1)^2 + (\vec{r}\cdot\hat{\epsilon}_2)^2 + (\vec{r}\cdot\hat{k})^2 = r^2$$

$$(\vec{r}\cdot\hat{\epsilon}_1)^2 + (\vec{r}\cdot\hat{\epsilon}_2)^2 = r^2 - (\vec{r}\cdot\hat{k})^2 = r^2\sin^2\theta$$

where $\theta \equiv$ angle between $\vec{r}$ and $\vec{k}$

from:

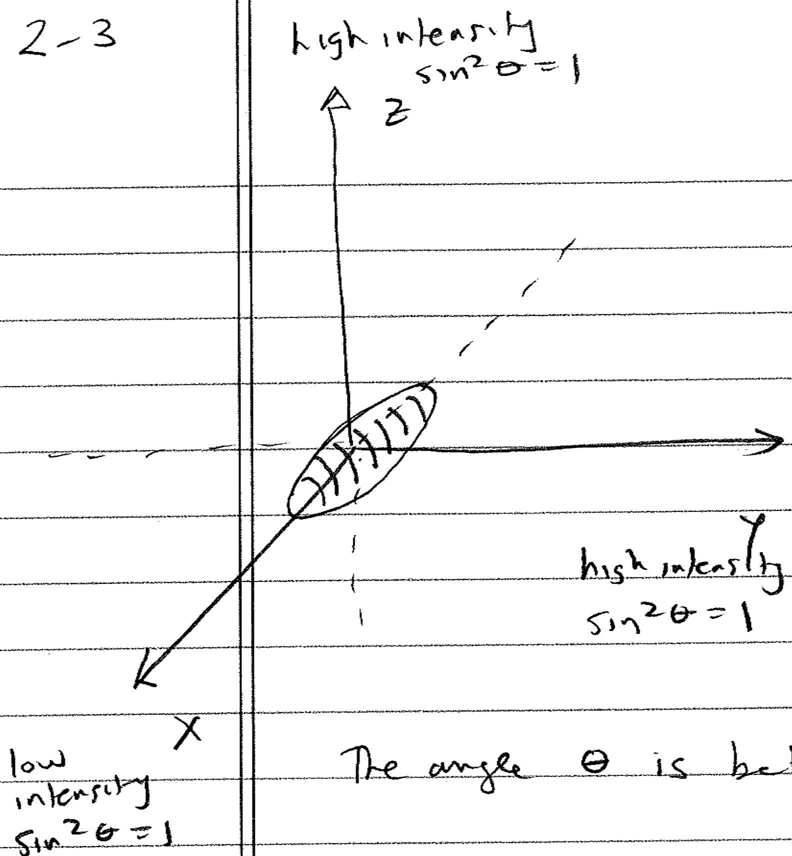
$$w_{e\to n} = \underbrace{\frac{2\pi}{\hbar}\frac{\omega^2}{(2\pi)^3}\frac{d\Omega}{\hbar c^3}}_{g(E)} \underbrace{\left(\frac{e}{mc}\right)^2}_{e/mc\,(\vec{p}\cdot\vec{A})} \underbrace{\left(c\sqrt{\frac{\hbar}{2\omega}}\right)^2}_{\text{units of } \vec{A}} \underbrace{(im\omega)^2}_{p\to[H_0,r] \text{ trick}}$$

$$\underbrace{\sum_\alpha |\langle n|\vec{r}\cdot\vec{\epsilon}_\alpha|e\rangle|^2}_{\sqrt{\frac{\hbar}{2m\omega}}\langle 000|a_x|100\rangle}$$

$n_x\, n_y\, n_z$

ground state — one of first excited states

2-3



Consider first excited state

$n_x = 1 \quad n_y = 0 \quad n_z = 0$

which schematically has greater extent in x-direction

The angle θ is between the emitted photon's direction $\hat{k}$ and the direction of the matrix element $\langle n | \vec{r} | e \rangle$

so we will get radiation in the y/z plane where $\sin^2\theta = 1$ but not much along the z axis where $\sin^2\theta = \phi$

Putting things together

$$w_{e\to n} = \frac{2\pi}{\hbar}\underbrace{\frac{\omega^2}{(2\pi)^3}\frac{d\Omega}{\hbar c^3}}_{g(E)}\left(\frac{e}{mc}\right)^2\left(c\sqrt{\frac{\hbar}{2\omega}}\right)^2(im\omega)^2$$

$\frac{e}{mc}(\vec{p}\cdot\vec{A})$ $\qquad A \qquad p \to [H_0, r]$ trick

$$\int \sum_\alpha |\langle n|\vec{r}\cdot\hat{\epsilon}_\alpha|e\rangle|^2 \, d\Omega$$

$$\left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2 \int_0^{2\pi} d\phi \int_0^\pi \sin^2\theta \sin\theta \, d\theta$$

$$2\pi \qquad 4/3$$

$$= \frac{\omega^2}{(2\pi)^2}\frac{1}{\cancel{\hbar^2} c^3}\frac{e^2}{\cancel{m^2}\cancel{c^2}}\cancel{c^2}\frac{\cancel{\hbar}}{2\cancel{\omega}}\cancel{m^2}\cancel{\omega^2}\frac{\cancel{\hbar}}{2m\cancel{\omega}}\frac{8\pi}{3}$$

$$= \frac{e^2\omega^2}{mc^3}\frac{1}{3\pi} \qquad \tau = \frac{3\pi mc^3}{e^2\omega^2}$$

I am not sure all factors of 2, π correct here...

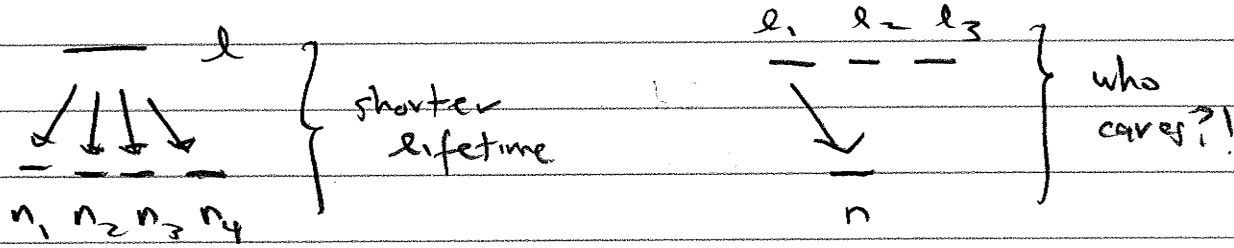
Units are correct! $\frac{[e^2]}{L^2} = \text{FORCE} = \frac{ML}{T^2}$

so $[e^2] = ML^3/T^2$ in cgs

$$\left[\frac{Mc^3}{e^2\omega^2}\right] = M\frac{L^3}{T^3}\frac{1}{\frac{ML^3}{T^2}\frac{1}{T^2}} = T \checkmark$$

2-5

My intuition says that degeneracy cannot matter. What is relevant (see Fermi's golden rule) is the number of states to which one can decay



but why would you care about having several degenerate states l_2, l_3 along with the one you are in?)

On the other hand, one might argue that by "spreading out" amongst many degenerate initial states one can get a larger matrix element

$\langle n|\vec{r}|l\rangle$

$= \langle 000|\ x\hat{x} + y\hat{y} + z\hat{z}\ \left(\alpha|100\rangle + \beta|010\rangle + \gamma|001\rangle\right)$

$= \sqrt{\frac{\hbar}{2m\omega}}\ (\alpha+\beta+\gamma)$ $\qquad |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$

so $|\langle n|\vec{r}|l\rangle|^2 = \frac{\hbar}{2m\omega}\ |\alpha+\beta+\gamma|^2$

2-6

Suppose, for example $\alpha = \beta = \gamma = \frac{1}{\sqrt{3}}$

which obeys $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$ so $|\ell\rangle$

is normalized.

we have $|\alpha + \beta + \gamma|^2 = 3$

so a. decay rate 3x higher and lifetime 3x

shorter.

I am not quite sure since this calculation

disagrees with my intuition that τ should not

be affected by degeneracy in the <u>initial</u> state.

Let's first get $f(\theta)$ using the formula in the notes

(which assumes spherically symmetric $V(\vec{r}) = V(r)$)

$$f(\theta) = \frac{-2m}{\hbar^2 q}\int_0^\infty r\,dr \left\{ -\frac{e^2 e^{-r/a_0}}{r} \sin qr \right\}$$

$$= \frac{2me^2}{\hbar^2 q}\frac{1}{2i}\int_0^\infty dr\, e^{-r/a_0}\left\{ e^{iqr} - e^{-iqr} \right\}$$

$$= \frac{me^2}{\hbar^2 q i}\left\{ \frac{e^{-r/a_0 + iqr}}{-1/a_0 + iq} - \frac{e^{-r/a_0 - iqr}}{-1/a_0 - iq} \right\}_0^\infty$$

$$= \frac{me^2}{\hbar^2 q i}\left\{ \frac{+1/a_0 + iq + 1/a_0 - iq}{1/a_0^2 + q^2} \right\}$$

$$= \frac{2me^2 a_0^2}{\hbar^2(1 + q^2 a_0^2)}$$

$a_0 = \hbar^2/me^2$

$q = 2k\sin\theta/2$

$$f(\theta) = \frac{2me^2 a_0^2}{\hbar^2}\left(1 + 4k^2 a_0^2 \sin^2\frac{\theta}{2}\right)^{-1}$$

$$d\sigma/d\Omega = |f|^2 = 4a_0^2\left(1 + 4k^2 a_0^2 \sin^2\frac{\theta}{2}\right)^{-2}$$

units of area!
("cross-section")

$d\sigma/d\Omega$ tells us about since scattering event

We have a lot of e^- coming in, and many H atoms

acting as possible targets

$$\frac{dN}{dt\,d\Omega} = \frac{dN_0}{dt}\;\frac{N}{A}\;\frac{d\sigma}{d\Omega}$$

Incident flux

\# targets per unit area

$$= 10^{-6}/1.6\cdot10^{-19} \qquad 10^{-3}\;6\cdot10^{23}\;1$$

$1\rho\left(\frac{g}{cm^3}\right)$; #H per gram ; thickness in cm

$$\frac{dN}{dt} = 4a_0^2\left(1+4k^2a_0^2\sin^2\frac{\theta}{2}\right)^{-2}\;\frac{10^{-6}}{1.6\cdot10^{-19}}\;10^{-3}\,6\cdot10^{23}\;\underbrace{10^{-5}}_{d\Omega}$$

$$K = \frac{\sqrt{2mE}}{\hbar} = \frac{\sqrt{2\cdot \underbrace{9.11\cdot10^{-28}}_{m}\;\underbrace{10^{5}}_{100\,keV}\;\underbrace{1.6\cdot10^{-12}}_{eV}}}{\underbrace{10^{-27}}_{\hbar}} = 1.71\cdot10^{10}\ cm^{-1}$$

$$a_0 = 5\cdot10^{-9}\ cm \qquad \Rightarrow ka_0 = 85$$

Meanwhile $\sin\frac{\theta}{2} = \sin\frac{1}{200} \sim \frac{1}{200}$

$\theta = 10$ mrad

3-3

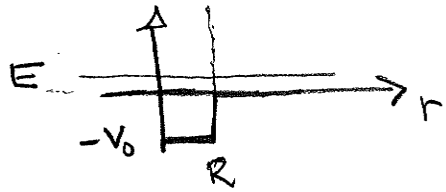
So $\left(1+4k^2a_0^2\sin^2\frac{\theta}{2}\right)^{-2}$ factor is

$$\left[1+4(85)^2\left(1/200\right)^2\right]^{-2} = (1.72)^{-2}$$

$$\frac{dN}{dt} = 4\underbrace{(5\cdot 10^{-9})^2}_{a_0^2}(1.72)^{-2}\ \underbrace{\frac{10^{13}}{1.6}}_{\text{incident flux}}\ \underbrace{6\cdot 10^{20}}_{\# H \text{ targets}}\ \underbrace{10^{-5}}_{d\Omega}$$

$$= 1.26\cdot 10^{17}\ e^-/\text{sec}.$$

4-1.



Radial Eqn

$$r<R \qquad \left\{\frac{d^2}{dr^2} + k_1^2 - \frac{l(l+1)}{r^2}\right\} u_l^{(1)}(r) = 0 \qquad k_1^2 = \frac{2m(E+V_0)}{\hbar}$$

$$r>R \qquad \left\{\frac{d^2}{dr^2} + k_2^2 - \frac{l(l+1)}{r^2}\right\} u_0^{(2)}(r) = 0 \qquad k_2^2 = \frac{2mE}{\hbar^2}$$

At low E only $l=0$ impt and eqn is just harmonic oscillator

$$u_0^{(1)} = A \sin k_1 r \qquad u_0^{(2)} = B \sin(k_2 r + \delta_0)$$

phase shift

Continuity of u_0 and its derivative at $r=R$:

$$\left.\frac{d/dr\, u_0^{(1)}}{u_0^{(1)}}\right|_R = \left.\frac{d/dr\, u_0^{(2)}}{u_0^{(2)}}\right|_R$$

$$\Rightarrow \qquad \frac{1}{k_1} \tan k_1 R = \frac{1}{k_2} \tan(k_2 R + \delta_0)$$

This determines phase shift

$$\delta_0 = \tan^{-1}\left(\frac{k_2}{k_1} \tan k_1 R\right) - k_2 R$$

4-2

Cross section is

$$\sigma = \frac{4\pi}{k_2^2} \sin^2 \delta_0$$

one can further simplify assuming E is small,

(1) If E small δ_0 will be small $\Rightarrow \sin\delta_0 \approx \delta_0$

(2) If E small $k_2 \to 0$ and $k_1 \to k_0 = \frac{2m V_0}{\hbar^2}$

(3) $\tan^{-1} x \approx x$

Using (1) $$\sigma = \frac{4\pi}{k_2^2}\left(\tan^{-1}\left(\frac{k_2}{k_1}\tan k_1 R\right) - k_2 R\right)^2$$

Using (2) + (3) $$= \frac{4\pi}{k_2^2}\left(\frac{k_2}{k_0}\tan k_0 R - k_2 R\right)^2$$

$$= 4\pi R^2 \left\{ \frac{\tan k_0 R}{k_0 R} - 1 \right\}^2$$

units of Area ✓✓

4-3

Large E use Born

$$f(\theta) = \frac{-2m}{\hbar^2 q}\int_0^\infty r\,dr\,V(r)\sin qr \qquad q = 2k\sin\frac{\theta}{2}$$

$$= \frac{-2m}{\hbar^2 q}\int_0^R \underset{\uparrow u}{r}\,\underbrace{dr\,V_0\sin qr}_{dv} \quad \Big\downarrow \text{ int by parts}$$

$$= \frac{-2mV_0}{\hbar^2 q}\left\{ -\frac{r\cos qr}{q}\Big|_0^R + \int_0^R \frac{\cos qr}{q}\,dr \right\}$$

$$= \frac{-2mV_0}{\hbar^2 q}\left\{ -\frac{R\cos qR}{q} + \frac{\sin qr}{q^2}\Big|_0^R \right\}$$

$$\uparrow$$
$$\sin qR/q^2$$

$$f(\theta) = \frac{-2mV_0R}{\hbar^2 q^2}\left\{ \frac{\sin qR}{qR} - \cos qR \right\}$$

$$\Big\downarrow \; q = 2k\sin\theta/2$$

$$\sigma = \int_0^{2\pi} d\phi \int_0^\pi \sin\theta\, d\theta\, |f(\theta)|^2$$

We are not asked to do integral.