Physics 215B- Quantum Mechanics, Winter 2014

Problem Set 2, due Tuesday January 28

[1.] What are the eigenvalues and eigenvectors of the tridiagonal matrix which has two different elements (alternating) along its diagonal? (Assume the matrix dimension N is even and that you have periodic boundary conditions.)

$$\hat{H} = \begin{pmatrix} A_1 & B & 0 & 0 & \cdots & 0 & B \\ B & A_2 & B & 0 & \cdots & 0 & 0 \\ 0 & B & A_1 & B & \cdots & 0 & 0 \\ 0 & 0 & B & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A_1 & B \\ B & 0 & 0 & 0 & \cdots & B & A_2 \end{pmatrix}$$

Discuss carefully the counting involved in your answer: Do you have the correct number of eigenvalues? Sketch the dependence of "energy" on "momentum". In what physics problem might such a matrix arise?

[2.] What does the matrix

$$\hat{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

do to a column vector with components $(\psi_1, \psi_2, \psi_3, \cdots \psi_N)$? Is there a reason I called this matrix "T"? What are its eigenvalues and eigenvectors? How are they related to those of the (simple) tridiagonal matrix discussed in class? Is that relation a surprise?

[3.] Consider a two site Heisenberg model in a Zeeman field.

$$\hat{H} = \frac{J}{\hbar^2} \hat{S}_1 \cdot \hat{S}_2 - \frac{B}{\hbar}(S_1^z + S_2^z)$$

What are the eigenvalues and eigenvectors? Plot the entanglement entropy of the ground state as a function of B for fixed $J = 1$. (Divide your system as in class so that $\mathcal{A}$ = spin 1 and $\mathcal{B}$ = spin 2.)

[4.] *(Qualifying Exam Problem!)* Consider a Hamiltonian of the form

$$\hat{H} = \frac{a}{\hbar^2} \hat{L}_z^2 + \frac{b}{\hbar} \hat{L}_x$$

where L_x and L_z are components of the orbital angular momentum of an atomic p-electron ($l = 1$). You may think of the first term in $\hat{H}$ as arising from a crystalline anisotropy which breaks the rotational symmetry and the second term as a magnetic field along the x axis. Write down the 3x3 matrix representation of $\hat{H}$. (You may want to use the formula

$L_\pm |lm\rangle = \hbar\sqrt{(l \mp m)(l \pm m + 1)}\,|l, m \pm 1\rangle$ where $L_\pm = L_x \pm iL_y$. Compute the energy eigenvalues and draw a rough graph showing the energy levels as functions of the field strength b.

[5.] *(Qualifying Exam Problem!)* Consider a very small lattice consisting of two atoms, each of which may be in either of two states. The wave function of each atom may be represented as a spin-1/2 system. The atoms interact with each other via the Hamiltonian

$$\hat{H}_1 = \Delta\, \hat{\sigma}_1^x\, \hat{\sigma}_2^x$$

In addition, the system is immersed in a magnetic field pointing in the z direction, giving a second term to the Hamiltonian,

$$\hat{H}_2 = \epsilon\, (\sigma_1^z + \sigma_2^z)$$

(The σ's are Pauli matrices.) What is the spectrum of the system under the combined Hamiltonian $\hat{H} = \hat{H}_1 + \hat{H}_2$? What are the eigenstates? How does the gap (difference between the ground and first excited state) behave as $\epsilon/\Delta \to 0, \infty$?

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Problem Set 2

$\boxed{1}$ We must decide on a good ansatz for the eigenvectors. It seems clear that our guess must differ depending on whether the component is odd or even. Let's allow them to have different amplitudes (but not different k!)

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \vdots \\ \psi_N \end{pmatrix} = \begin{pmatrix} ue^{ik} \\ ve^{ik} \\ ue^{2ik} \\ ve^{2ik} \\ \vdots \\ ve^{N/2\,ik} \end{pmatrix}$$

Notice we are also assuming the two components $(1,2)$, $(3,4)$, $(5,6)$, ... have the same phase e^{ik}, e^{2ik}, ...

$$\left.\begin{aligned} \psi_{2\ell-1} &= ue^{ik\ell} \\ \psi_{2\ell} &= ve^{ik\ell} \end{aligned}\right\} \ell = 1,2,\ldots N/2$$

(We will come back to different guesses later...)

$$A_1 \psi_{2\ell+1} + B\psi_{2\ell} + B\psi_{2\ell+2} = E\psi_{2\ell+1}$$

$$A_2 \psi_{2\ell} + B\psi_{2\ell-1} + B\psi_{2\ell+1} = E\psi_{2\ell}$$

Substituting our assumed form

cancel $e^{ik(\ell+1)}$ → $A_1 u e^{ik(\ell+1)} + Bve^{ik\ell} + Bve^{ik(\ell+1)} = Eue^{ik(\ell+1)}$

cancel $e^{ik\ell}$ → $A_2 ve^{ik\ell} + Bue^{ik\ell} + Bue^{ik(\ell+1)} = Eve^{ik\ell}$

$$\Rightarrow \begin{bmatrix} A_1 - E & B(1+e^{-ik}) \\ B(1+e^{ik}) & A_2 - E \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Determinant must vanish for non trivial soln:

$$\underbrace{(A_1 - E)(A_2 - E)}_{E^2 - (A_1+A_2)E + A_1A_2} - B^2 \underbrace{(2 + 2\cos k)}_{= 2(1+\cos k) = 4\cos^2 k/2} = 0$$

$$E = \frac{1}{2}\left\{ (A_1 + A_2) \pm \sqrt{(A_1+A_2)^2 - 4(A_1A_2 - 4\underset{B^2}{\cos^2}\frac{k}{2})} \right\}$$

$$= \frac{1}{2}\left\{ (A_1 + A_2) \pm \sqrt{(A_1 - A_2)^2 + 16B^2\cos^2\frac{k}{2}} \right\}$$

There are 2 eigenvalues $E_\pm$ for each k value! How many allowed k's are there?

To determine allowed k's we look at "bdy equations"

$$A_1\psi_1 + B\psi_2 + B\psi_0 = E\psi_1$$

was our <u>assumed</u> form, but really the eqn $M\vec{\psi} = E\vec{\psi}$

has $A_1\psi_1 + B\psi_2 + B\psi_N = E\psi_1$

We must require $\psi_0 = \psi_N$.

Plugging in $\psi_{2\ell} = v e^{ik\ell}$ (assume N is even)

$$v = v e^{ikN/2}$$

Thus $e^{ikN/2} = 1$

$$k = \frac{2\pi}{N/2}\{1, 2, \ldots N/2\} = \frac{4\pi}{N}\{1, 2, \ldots \tfrac{N}{2}\}$$

There are $\frac{N}{2}$ allowed k values and 2 eigenvalues $E_\pm$ for each k.

The eigenvectors have the form of page 1-1

except we might want explicit values for u, v,

These are given by

$$(A_1 - E)u + B(1+e^{-ik})v = 0$$

i.e. $v/u = \dfrac{E - A_1}{B(1+e^{-ik})}$

Note there is nothing wrong with v, u being complex,

only eigenvalues are guaranteed to be real.

So this is the end of the story. There

have been some questions about the problem which I

will now address...

It is almost always useful to check solutions for small matrices. Let's try $N=4$

$$M = \begin{pmatrix} A_1 & B & 0 & B \\ B & A_2 & B & 0 \\ 0 & B & A_1 & B \\ B & 0 & B & A_2 \end{pmatrix}$$

The $k=\pi$ eigenvalues are just $E=A_1$ and $E=A_2$

(Actually you can see the choice $E=A_1$ makes rows 1, 3 of the matrix linearly dependent!)

The values of u and v (see page 1-2)

are $u=1$ $v=0$ or $u=0$ $v=1$. Does this work?

That is Are $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} u \\ v \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ eigenvectors of M?

$k=\pi$ means second $\begin{pmatrix} u \\ v \end{pmatrix}$ set has $e^{i\pi}$ relative to first

These clearly work! whew!

In fact, these clearly work also for larger N!

How about $k = 2\pi$? This is less simple.

The eigenvalues are $\frac{1}{2}\left\{ (A_1 + A_2) \pm \sqrt{(A_1 - A_2)^2 + 16B^2} \right\}$

and the eigenvectors $\begin{pmatrix} u \\ v \end{pmatrix}$ are non-trivial.

All we can examine easily is whether $\begin{pmatrix} u \\ v \\ u \\ v \\ \vdots \end{pmatrix}$

makes sense

$$M \begin{pmatrix} u \\ v \\ u \\ v \\ \vdots \end{pmatrix} = \begin{pmatrix} A_1 u + 2Bv \\ A_2 v + 2Bu \\ A_1 u + 2Bv \\ A_2 v + 2Bu \\ \vdots \end{pmatrix} \stackrel{?}{=} E \begin{pmatrix} u \\ v \\ u \\ v \\ \vdots \end{pmatrix}$$

What we see is that the N eqns are actually

only 2 eqns
$$A_1 u + 2Bv = Eu$$
$$A_2 v + 2Bu = Ev$$

and this is the $k=0$ form of the eqn for $\begin{pmatrix} u \\ v \end{pmatrix}$

on page 1-2. So we have not done much

except seen things are consistent

1-5.

(A) Some of you have looked at the relative sign of u and v. For example, let's examine $k=0$ where $e^{-ik}=1$ and pick $A_1=8$ $A_2=6$ $B=1$ just as an example. Then

$$E_+ = \frac{1}{2}\left\{14 + \sqrt{4+16}\right\} = 9.24$$

$$E_- = \frac{1}{2}\left\{14 - \sqrt{4+16}\right\} = 4.76$$

$$(v/u)_+ = (9.24-8)/2 = +0.62$$

$$(v/u)_- = (4.76-8)/2 = -1.62$$

This surprised some of you who have experience in solid state physics and expected the higher energy mode to have odd, even "out of phase" ($v/u < 0$)

We find the opposite. This is $\underline{\underline{\text{not}}}$ a problem. The reason is that this problem is not precisely the same as the two mass M_1, M_2 optic/acoustic phonon problem in condensed matter physics...

1-6

Whether the largest / smallest λ is associated with in phase / out of phase depends on signs in matrix, Eg,

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad E_1 = 3 \quad \psi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$E_2 = 1 \quad \psi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad E_1 = 1 \quad \psi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$E_2 = 3 \quad \psi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

As you can see, relative sign of eigenvector components of maximal E depends on signs in M.

So don't worry about the sign of our v/u ratio differing from your condensed matter physics intuition! The CMP problem has different signs.

Put another way if I had made $A_1 = 8$ $A_2 = 6$ $B = -1$ I would have found the "expected"

$$(v/u)_+ < 0 \qquad (v/u)_- > 0$$

1-7

(B) One might also wonder if an assumption more like that in class for ψ_ℓ would have worked. That is, could we have assumed

$$\psi_\ell = u e^{ik\ell} \qquad \ell \text{ odd}$$
$$= v e^{ik\ell} \qquad \ell \text{ even}$$

or, writing it out explicitly,

$$\psi = \begin{pmatrix} u e^{ik} \\ v e^{2ik} \\ u e^{3ik} \\ v e^{4ik} \\ \vdots \end{pmatrix}$$

Could we have assumed this form?!

Lets try! For odd/even rows

$$A_1 u e^{ik\ell} + B v e^{ik(\ell-1)} + B v e^{ik(\ell+1)} = E u e^{ik\ell}$$

$$A_2 v e^{ik\ell} + B u e^{ik(\ell-1)} + B u e^{ik(\ell+1)} = E v e^{ik\ell}$$

1-8

$$A_1 u + 2B\cos k\, v = Eu$$

$$A_2 v + 2B\cos k\, u = Ev$$

$$\begin{pmatrix} A_1 - E & 2B\cos k \\ 2B\cos k & A_2 - E \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad (A_1 - E)(A_2 - E) - 4B^2\cos^2 k = 0$$

This is looks a lot like our previous eqn (p1-2)

$$(A_1 - E)(A_2 - E) - 4B^2\cos^2\frac{k}{2} = 0$$

So it looks like the two methods are giving the same answer, except we need to convince ourselves that the allowed k values agree. In our "old" method we had $k = \frac{4\pi}{N}\{1, 2, \ldots N/2\}$ and $\cos^2\frac{k}{2}$ in the eigenvalue eqn, In our new method we have $k = \frac{2\pi}{N}\{1, 2, \ldots N\}$ since boundaries demand $\psi_1 = \psi_{N+1}$

k's are half as big, but that is compensated by $\cos^2\frac{k}{2} \to \cos^2 k$

so we learn the 2 ansatz give exact same eigenvalues! Do it either way!

The only remaining puzzle is getting the counting correct in the "new" method. It looks as if we have N k values and $2N$ Eigenvalues E.

The answer to this dilemma is that some of the eigenvalues/eigenvectors are duplicates!

I'll let you track this down,

2 Clearly

$$\hat{T}\begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\\ \\ \psi_N\end{pmatrix} = \begin{pmatrix}\psi_2\\ \psi_3\\ \vdots\\ \\ \psi_1\end{pmatrix}$$

$\hat{T}$ "translates" the components of $\vec{\psi}$ by 1 space, hence its name. Writing the eigenequations in component form

$$\begin{aligned}\psi_2 &= \lambda\psi_1 \\ \psi_3 &= \lambda\psi_2 \qquad \psi_3 = \lambda^2\psi_1 \qquad \psi_4 = \lambda^3\psi_1 \\ &\vdots \\ \psi_N &= \lambda\psi_{N-1} \\ \psi_1 &= \lambda\psi_N \qquad \psi_1 = \lambda^N\psi_1\end{aligned}$$

Thus $\lambda^N = 1$ and $\lambda = e^{ik}$ $K = \frac{2\pi}{N}\{1,2,3,..N\}$

Setting $\psi_1 = e^{ik}$ we see $\psi_2 = e^{2ik}$

and in general $\psi_\ell = e^{ik\ell}$

same eigenvectors as our (simple) tridiagonal matrix $A_1 = A_2$ version of Problem 1

$$\begin{pmatrix}A & B & & \\ B & A & B & \\ & B & A & B \\ & & & \ddots\end{pmatrix}$$

This relation is not a surprise since

$\hat{T}$ commutes with the (simple) tridiagonal matrix,

so they share eigenvectors.

Actually $\hat{T}$ commutes with <u>any</u> matrix

with the form obtained by shifting elements of rows over by 1 step.

$$\begin{bmatrix} A & B & C & D & E & \dots & Z \\ Z & A & B & C & D & \dots & \\ Y & Z & A & B & C & \dots & \\ & & & & & & \\ & & & & & & \end{bmatrix}$$

A deeper comment is to recall that in QM

the translation operator $\hat{T} = e^{i\Delta x \hat{p}/\hbar}$ because

$$\hat{T}\psi(x) = e^{\Delta x \, d/dx}\psi(x) = \left[1 + \Delta x \frac{d}{dx} + \frac{1}{2}(\Delta x)^2 \frac{d^2}{dx^2} + \dots\right]\psi(x)$$

$$= \psi(x) + \Delta x \frac{d\psi}{dx} + \frac{1}{2}(\Delta x)^2 \frac{d^2}{dx^2}\psi(x) + \dots$$

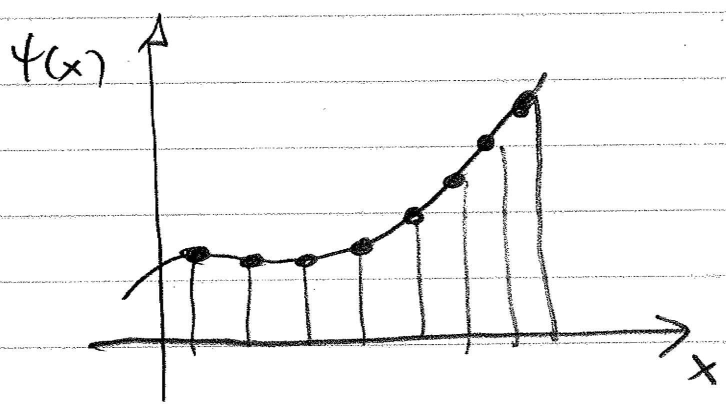
$$= \psi(x + \Delta x)$$

by Taylor's Theorem.

2-3

If we imagined storing $\psi(x)$ as a vector on discrete grid points $\begin{pmatrix} \psi(\Delta x) \\ \psi(2\Delta x) \\ \psi(3\Delta x) \\ \vdots \end{pmatrix}$

Then clearly translation by Δx would be accomplished by acting with $\hat{T}$



$\boxed{3.}$ Writing out $\vec{S}_1 \cdot \vec{S}_2$ we have

$$\hat{H} = \frac{J}{\hbar^2} S_1^z S_2^z + \frac{J}{2\hbar^2}\left(S_1^+ S_2^- + S_1^- S_2^+\right) - \frac{B}{\hbar}\left(S_1^z + S_2^z\right)$$

Acting on the four basis vectors $|S_1^z S_2^z\rangle$ gives

$$\hat{H}|++\rangle = \left(J/4 - B\right)|++\rangle$$

$$\hat{H}|--\rangle = \left(J/4 + B\right)|--\rangle$$

$$\hat{H}|+-\rangle = -J/4\,|+-\rangle + J/2\,|-+\rangle$$

$$\hat{H}|-+\rangle = -J/4\,|-+\rangle + J/2\,|+-\rangle$$

The eigenvalues and eigenvectors of $\hat{H}$ are

$-3J/4$	$1/\sqrt{2}\left(|+-\rangle - |-+\rangle\right)$
$+J/4 - B$	$|++\rangle$
$J/4$	$1/\sqrt{2}\left(|+-\rangle + |-+\rangle\right)$
$J/4 + B$	$|--\rangle$

The field B splits the degenerate triplet into 3 distinct eigenvalues.

We computed the entanglement entropy of

$$\frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

in class. This is the ground state as long as $B < J$

When $B > J$ the ground state becomes $|++\rangle$

(spins align with field to lower energy)

The entanglement entropy is computed by writing

the full density matrix $\hat{\rho} = |++\rangle\langle++|$

or, if we order basis vectors $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$

as we have been doing in class

$$\hat{\rho} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The reduced density matrix $\hat{\rho}_A$ is obtained by

tracing over the B (spin 2) degrees of freedom.

3-3

$$\overset{\downarrow S_1^z}{\langle +|}\hat{\rho}_A|+\rangle = \langle ++|\hat{\rho}|++\rangle + \overset{\downarrow S_1^{z\prime}}{\langle +-|}\hat{\rho}|+-\rangle = 1+0 = 1$$

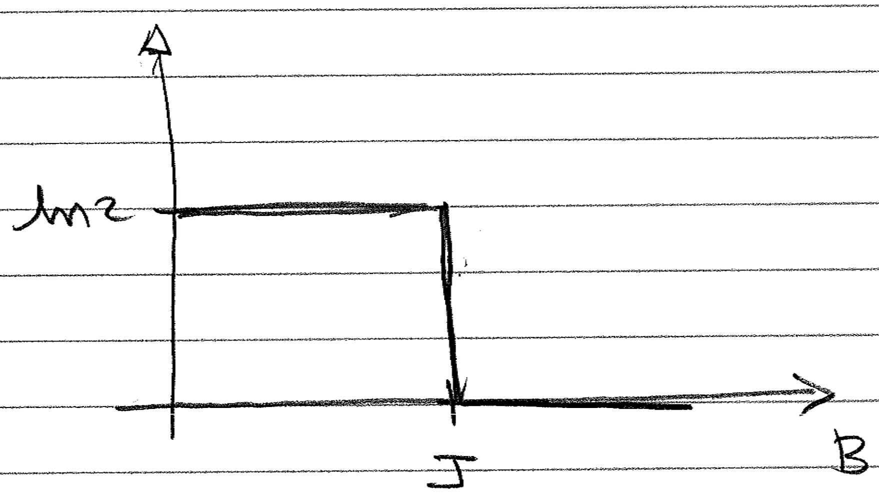
$$\langle +|\hat{\rho}_A|-\rangle = \langle ++|\hat{\rho}|-+\rangle + \langle +-|\hat{\rho}|--\rangle = 0+0 = 0$$

$$\langle -|\hat{\rho}_A|+\rangle = \qquad\qquad = 0$$

$$\langle -|\hat{\rho}_A|-\rangle = \qquad\qquad = 0$$

$$\hat{\rho}_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$ ($\hat{\rho}_A^2 = \hat{\rho}_A$ as expected for pure state)

$$S_A = -\ln tr\, \hat{\rho}_A^2 = -\ln tr \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = -\ln 1 = \phi$$



S_A has a sharp change when B goes through J.

The conjecture is that S_A in general provides a signature of phase transitions in more general problems (Big lattices).

$\boxed{4}$ Choose as a basis $|1\rangle$ $|0\rangle$ $|-1\rangle$

ie states $|\ell m\rangle$ with $\ell = 1$ and $m = 1, 0, -1$

We have $\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)$ so

$$\hat{H}|1\rangle = a|1\rangle + \tfrac{1}{2} b\sqrt{2}|0\rangle$$

$$\hat{H}|0\rangle = \tfrac{1}{2} b\sqrt{2}|1\rangle + \tfrac{1}{2} b\sqrt{2}|-1\rangle$$

$$\hat{H}|-1\rangle = a|-1\rangle + \tfrac{1}{2} b\sqrt{2}|0\rangle$$

$$\hat{H} = \begin{pmatrix} a & b\sqrt{2}/2 & 0 \\ b\sqrt{2}/2 & 0 & b\sqrt{2}/2 \\ 0 & b\sqrt{2}/2 & a \end{pmatrix}$$

Eigenvalues are

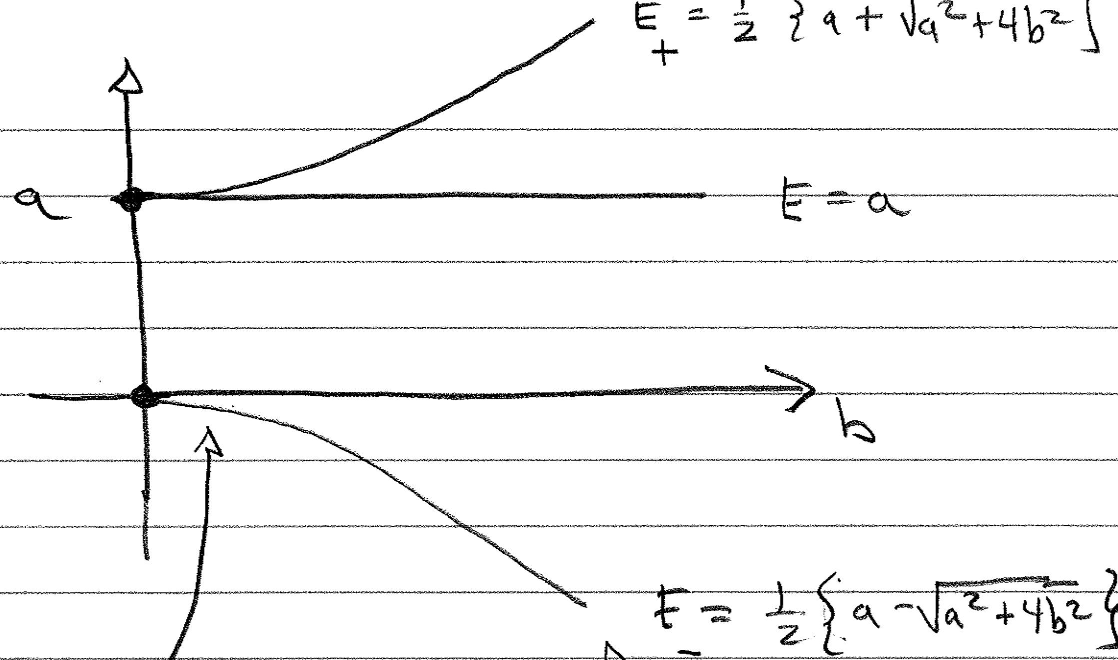
$$(a-E)\left[-E(a-E) - b^2/2\right] - b\sqrt{2}/2 \; b\sqrt{2}/2 \, (a-E) = 0$$

$$(a-E)(E^2 - Ea - b^2/2) - b^2/2 \, (a-E) = 0$$

$$(a-E)\left[E^2 - Ea - b^2\right] = 0$$

$$E = a \qquad E = \frac{1}{2}\left\{a \pm \sqrt{a^2 + 4b^2}\right\}$$

4-2



quadratic in b for b small

$$E_- = \frac{1}{2}\left(a - a\left(1 + 4b^2/a^2\right)^{1/2}\right)$$

$$\approx \frac{1}{2}\left(a - a\left(1 + 2b^2/a^2 + \dots\right)\right)$$

$$= -2b^2/a$$

linear in b for large b

$$E_- \approx -b$$

5-1

(I should have used $\hat{H}_1 = \Delta/\hbar^2\, \hat{\sigma}_1^x \hat{\sigma}_2^x$; $\hat{H}_2 = \epsilon/\hbar\, (\hat{\sigma}_1^z + \hat{\sigma}_2^z)$)

$\boxed{5}$ The basis is $|\sigma_1^z \sigma_2^z\rangle$ and we can rewrite $\hat{H}$ more conveniently as

$$\hat{H} = \frac{\Delta}{4}(\sigma_1^+ + \sigma_1^-)(\sigma_2^+ + \sigma_2^-) + \epsilon(\sigma_1^z + \sigma_2^z)$$

Then

$$\hat{H}|++\rangle = \frac{\Delta\hbar^2}{4}|--\rangle + \epsilon\hbar\,|++\rangle$$

$$\hat{H}\,|+-\rangle = \frac{\Delta\hbar^2}{4}|-+\rangle$$

$$\hat{H}\,|-+\rangle = \frac{\Delta\hbar^2}{4}|+-\rangle$$

$$\hat{H}\,|--\rangle = \frac{\Delta\hbar^2}{4}|++\rangle - \epsilon\hbar\,|--\rangle$$

Eigenvalues are those of 2 indep 2x2 blocks

$$\begin{array}{c} |++\rangle \\ |--\rangle \\ |+-\rangle \\ |-+\rangle \end{array} \begin{pmatrix} \epsilon\hbar & \Delta\hbar^2/4 & 0 & 0 \\ \Delta\hbar^2/4 & -\epsilon\hbar & 0 & 0 \\ 0 & 0 & 0 & \Delta\hbar^2/4 \\ 0 & 0 & \Delta\hbar^2/4 & 0 \end{pmatrix}$$

$$E = \pm\, \Delta\hbar^2/4$$

$$E = \pm\sqrt{(\Delta\hbar^2/4)^2 + \epsilon^2\hbar^2}$$ since $(\epsilon\hbar - E)(-\epsilon\hbar - E) - \left(\frac{\Delta\hbar^2}{4}\right)^2$

$= +E^2 - \epsilon^2\hbar^2 - \left(\frac{\Delta\hbar^2}{4}\right)^2 = 0$

5-2

The eigenvectors for $E = \pm \Delta\hbar^2/4$ are simple:

$$+\frac{\Delta\hbar^2}{4} \longrightarrow \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)$$

$$-\frac{\Delta\hbar^2}{4} \longrightarrow \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

The eigenvectors of $E_\pm = \pm\sqrt{(\Delta\hbar^2/4)^2 + \epsilon^2\hbar^2}$

are more complicated eg

$$(\epsilon\hbar - E_+)\, a + \Delta\hbar^2/4\, b = 0$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{4(\epsilon\hbar - E_+)}{\Delta\hbar^2} \end{pmatrix} \longleftarrow \text{unnormalized}$$

divide by $\left\{1 + \left(\frac{4(\epsilon\hbar - E_+)}{\Delta\hbar^2}\right)^2\right\}^{1/2}$

to normalize.

Similarly for E_-.