Physics 215B- Quantum Mechanics, Winter 2014

Problem Set 1, due Thursday January 16

The first two problems review some of Physics 215A.

[1.] Consider a system whose state space, which is three dimensional, is spanned by the orthonormal basis $|u_1\rangle, |u_2\rangle, |u_3\rangle$. In this basis, the Hamiltonian H and the observable M are given by,

$$\hat{H} = E \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \hat{M} = F \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

Here E and F are real numbers. The initial state is

$$|\Psi(t=0)\rangle = \frac{3}{5}|u_2\rangle + \frac{4}{5}|u_3\rangle.$$

What is the expectation value of the energy, $\langle \hat{H} \rangle$ at $t = 0$?

What are the possible values of a measurement of $\hat{M}$?

What is the expectation value $\langle \hat{M} \rangle(t)$ at arbitrary time t?

[2.] Re–do problem 1 with the matrices for $\hat{H}$ and $\hat{M}$ interchanged.

[3.] We solved the $N = 2$ site (spin-1/2) Heisenberg Hamiltonian in class both by 'brute force' (explicit construction and diagonalization) and also by a 'trick' using the rules for adding angular momentum. Do the same 'brute force' solution of the three site case $\hat{H} = J/\hbar^2 \, (\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_1 \,)$. Can you also do the case where spins 1 and 3 are not connected: $\hat{H} = J/\hbar^2 \, (\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 \,)$?

[4.] We also solved the $N = 4$ site problem in class using the 'trick'. What are the eigenenergies with an additional 'frustrating' interaction J':

$$\hat{H} = J/\hbar^2 \, (\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_4 + \vec{S}_4 \cdot \vec{S}_1 \,) + J'/\hbar^2 \, (\vec{S}_1 \cdot \vec{S}_3 + \vec{S}_2 \cdot \vec{S}_4 \,)$$

Discuss what happens as J'/J increases. We will talk further about this in class after you have had a chance to think about the problem independently. This problem provides a simple illustration of a 'quantum phase transition'.

[5.] How big is the Hilbert space for the spin-1/2 Heisenberg Hamiltonian with N spins? Describe how you might write a program to set up the matrix for $\hat{H}$. (If you could do this, you could then call a matrix diagonalization library routine and thereby completely solve the problem numerically.) I don't want an actual working code in any specific language, but rather your thoughts on the algorithmic steps which would be useful/necessary. Discuss how large an N is feasible to do with a workstation. (How much memory is required for a given N? How much cpu time?)

[6.] (Optional) Set up $\hat{H}$ and diagonalize for the $N = 4$ spin J-J' Heisenberg model and check the eigenvalues you obtained in Problem 4. How much of it can you do with pencil and paper? Can you do the rest with a computer?

1.

Physics 215B Winter 2013
HW1 Sol'ns

$\boxed{1}$ $\hat{H}$ is diagonal so $|u_i\rangle$ are eigenvectors.

Can immediately write $|\psi(t)\rangle = \frac{3}{5} e^{-i\omega t} |u_2\rangle + \frac{4}{5} e^{-2i\omega t} |u_3\rangle$

with $\omega \equiv E/\hbar$.

$$\langle \hat{H} \rangle (t=0) = \frac{1}{5} \begin{pmatrix} 0 & 3 & 4 \end{pmatrix} E \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$$

$$= E/_{25} \begin{pmatrix} 0 & 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 8 \end{pmatrix} = 41/_{25} E$$

Possible values for measuring M are eigenvalues of $\hat{M}$:

$-2F, +F, +2F$

$$\langle \hat{M} \rangle (t) = \langle \psi(t) | \hat{M} | \psi(t) \rangle$$

$$= \frac{1}{5} \begin{pmatrix} 0 & 3e^{+i\omega t} & 4e^{+2i\omega t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cancel{0} & 2 \\ 0 & 2 & 0 \end{pmatrix} F \frac{1}{5} \begin{pmatrix} 0 \\ 3e^{-i\omega t} \\ 4e^{-2i\omega t} \end{pmatrix}$$

$$= \frac{F}{25} \left(\qquad \right) \begin{pmatrix} 0 \\ 8e^{-2i\omega t} \\ 6e^{-i\omega t} \end{pmatrix}$$

$$= \frac{F}{25} \left(24 e^{-i\omega t} + 24 e^{+i\omega t} \right) = \frac{48F}{25} \cos \omega t$$

2.

$\boxed{2.}$ Now $\hat{H} = F\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$ $\quad \hat{M} = E\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$\langle \hat{H} \rangle (t=0) = \frac{1}{5}\begin{pmatrix} 0 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} F \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} 1/5$$

$$= \frac{F}{25}\begin{pmatrix} 0 & 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} = \frac{F}{25} 48.$$

Possible values of M measurement are eigenvalues $E, E, 2E$

Eigenstates of $\hat{H}$ are $|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad |e_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\quad |e_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$E_1 = F \qquad E_2 = 2F \qquad E_3 = -2F$

$$|\psi(t)\rangle = \sum_n e^{-i\omega_n t} |e_n\rangle \langle e_n | \psi(t=0)\rangle \qquad \omega_n = E_n/\hbar$$

Define $\omega = F/\hbar$

$$= e^{-i\omega t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \phi + e^{-2i\omega t} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} \frac{7}{5} + e^{+2i\omega t} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}}\left(\frac{-1}{5}\right)$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 3/5 \\ 4/5 \end{pmatrix}$$

$$|\psi(t)\rangle = \left(\frac{7}{10} e^{-2i\omega t} - \frac{1}{10} e^{2i\omega t}\right)|u_2\rangle + \left(\frac{7}{10} e^{-2i\omega t} + \frac{1}{10} e^{2i\omega t}\right)|u_3\rangle$$

3.

2 //cont'd

$$\langle \hat{M} \rangle (t) = \langle \psi(t) | \hat{M} | \psi(t) \rangle$$

$$= \frac{1}{10} \begin{pmatrix} 0 & 7e^{+2i\omega t} - e^{-2i\omega t} & 7e^{+2i\omega t} + e^{-2i\omega t} \end{pmatrix} E \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 7e^{-2i\omega t} - e^{2i\omega t} \\ 7e^{-2i\omega t} + e^{+2i\omega t} \end{pmatrix} \frac{1}{10}$$

$$= \frac{E}{100} \Big\{ (7e^{2i\omega t} - e^{-2i\omega t}) 1 (7e^{-2i\omega t} - e^{2i\omega t})$$

$$+ (7e^{2i\omega t} + e^{-2i\omega t}) 2 (7e^{-2i\omega t} + e^{2i\omega t}) \Big\}$$

$$= \frac{E}{100} \Big\{ 49 - 7e^{4i\omega t} - 7e^{-4i\omega t} + 1$$

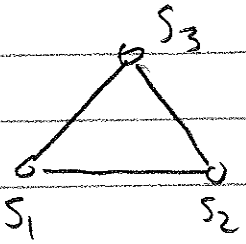
$$+ 98 + 14e^{-4i\omega t} + 14e^{4i\omega t} + 2 \Big\}$$

$$= \frac{E}{100} \Big\{ 150 + 14 \cos 4\omega t \Big\}$$

$$\langle \hat{M}(t) \rangle = 1.5E + \frac{7}{50} \cos 4\omega t$$

4.

$\boxed{3.}$



$$\hat{H} = \frac{J}{\hbar^2}\left(\vec{S}_1\cdot\vec{S}_2 + \vec{S}_2\cdot\vec{S}_3 + \vec{S}_3\cdot\vec{S}_1\right)$$

$$S_1^z S_2^z + \tfrac{1}{2}\left(S_1^+ S_2^- + S_1^- S_2^+\right)$$

Basis $\quad |S_1^z\ S_2^z\ S_3^z\rangle$

$$\hat{H}|+++\rangle = J\frac{3}{4}|+++\rangle$$

$$\hat{H}|++-\rangle = -\frac{J}{4}|++-\rangle + \frac{J}{2}|-++\rangle + \frac{J}{2}|+-+\rangle$$

Do rest in head $\quad$ Matrix of $\hat{H}$

$$\begin{array}{c} |+++\rangle \\ |++-\rangle \\ |+-+\rangle \\ |-++\rangle \\ |--+\rangle \\ |-+-\rangle \\ |+--\rangle \\ |---\rangle \end{array}
\begin{bmatrix}
3/4J & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -J/4 & J/2 & J/2 & 0 & 0 & 0 & 0 \\
0 & J/2 & -J/4 & J/2 & 0 & 0 & 0 & 0 \\
0 & J/2 & J/2 & -J/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -J/4 & J/2 & J/2 & 0 \\
0 & 0 & 0 & 0 & J/2 & -J/4 & J/2 & 0 \\
0 & 0 & 0 & 0 & J/2 & J/2 & -J/4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3J/4
\end{bmatrix}$$

Need eigenvalues of 3x3 matrix

$$J/4\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

5.

3// cont'd Get eigenvalues of $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} J/4$

and subtract $J/4$ at end. By inspection $\lambda = -2$

works (and is doubly degenerate). But do it the long way:

$$-\lambda(\lambda^2 - 4) - 2(-2\lambda - 4) + 2(4 + 2\lambda) = 0$$

$$-\lambda^3 + 4\lambda + 4\lambda + 8 + 8 + 4\lambda = 0$$

$$\lambda^3 - 12\lambda - 16 = 0$$

$$(\lambda + 2)^2(\lambda - 4) = 0 \qquad \lambda = -2, -2, 4$$

So eigenvalues are $J/4\,(-2, -2, 4) \underbrace{- J/4}_{\text{diagonal shift}} =$

Complete list of eigenvalues: $\frac{3J}{4}$ $-\frac{3J}{4}, -\frac{3J}{4}, \frac{3J}{4}$

$\frac{3J}{4}$: $|{+}{+}{+}\rangle$ and $|{-}{-}{-}\rangle$

$-\frac{3J}{4}$: $|{+}{+}{-}\rangle$, $|{+}{-}{+}\rangle$, $|{-}{+}{+}\rangle$ and same for $|{-}{-}{+}\rangle$, $|{-}{+}{-}\rangle$, $|{+}{-}{-}\rangle$

$\Rightarrow$ There are only 2 eigenvalues: $\pm 3J/4$ each 4-fold degenerate

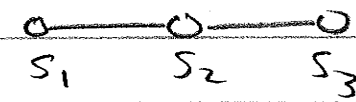
Eigenvectors include $|{+}{+}{+}\rangle$; $|{-}{-}{-}\rangle$ for $\lambda = 3J/4$

$\frac{1}{\sqrt{3}}\, |{+}{+}{-}\rangle + |{+}{-}{+}\rangle + |{-}{+}{+}\rangle$ for $\lambda = -3J/4$

etc.

3/ cont'd   Do the "open boundary case"

$$\mathcal{H} = \frac{J}{\hbar^2}\left(\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3\right)$$



$$\hat{H}|{+}{+}{+}\rangle = J\tfrac{1}{2}|{+}{+}{+}\rangle$$

$$\hat{H}|{+}{+}{-}\rangle = \frac{J}{2}|{+}{-}{+}\rangle$$

Do rest in head to get matrix of $\hat{H}$

$$\begin{array}{l} |{+}{+}{+}\rangle \\ |{+}{+}{-}\rangle \\ |{+}{-}{+}\rangle \\ |{-}{+}{+}\rangle \\ |{-}{-}{+}\rangle \\ |{-}{+}{-}\rangle \\ |{+}{-}{-}\rangle \\ |{-}{-}{-}\rangle \end{array}
\left(\begin{array}{c|ccc|ccc|c}
\frac{1}{2}J & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & \frac{1}{2}J & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2}J & -\frac{1}{2}J & \frac{1}{2}J & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2}J & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & \frac{1}{2}J & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2}J & -\frac{1}{2}J & \frac{1}{2}J & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}J & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}J
\end{array}\right)$$

Need eigenvalues of $\frac{J}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$-\lambda\left[(-\lambda-1)(-\lambda)-1\right] - \left[-\lambda - 0\right]$$
$$= -\lambda\left[\lambda^2+\lambda-1\right] + \lambda$$
$$= -\lambda^3 - \lambda^2 + 2\lambda = -\lambda\left(\lambda^2+\lambda-2\right) = -\lambda(\lambda+2)(\lambda-1)$$

7.

Eigenvalues are $\frac{J}{2}\underbrace{\{-2, 0, 1, 1\}}_{\text{each 2 fold degenerate}}$

NB, A check on eigenvalues is that their sum should equal the trace of original matrix. This works!

Eigenvectors are $|{+}{+}{+}\rangle$ and $|{-}{-}{-}\rangle$ (eigenvalue $\frac{J}{2}$)

Also $\frac{1}{\sqrt{2}}\left(|{+}{+}{-}\rangle - |{-}{+}{+}\rangle\right)$ (eigenvalue ϕ)

$\frac{1}{\sqrt{2}}\left(|{-}{-}{+}\rangle - |{+}{-}{-}\rangle\right)$

The other vectors of eigenvalue $J/2$ determined by

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x = y = z$$

So $\frac{1}{\sqrt{3}}\left(|{+}{+}{-}\rangle + |{+}{-}{+}\rangle + |{-}{+}{+}\rangle\right)$ Eigenvalue $\frac{J}{2}$

and $\frac{1}{\sqrt{3}}\left(|{-}{-}{+}\rangle + |{-}{+}{-}\rangle + |{+}{-}{-}\rangle\right)$

Finally $-J$ $$\begin{pmatrix} +2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow y = -2x = -2z$$

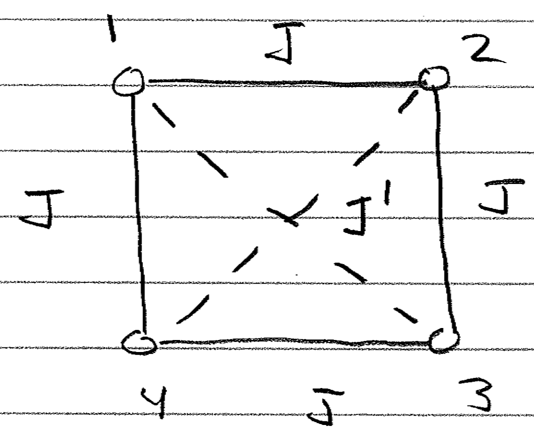
So $\frac{1}{\sqrt{6}}\left(|{+}{+}{-}\rangle - 2|{+}{-}{+}\rangle + |{-}{+}{+}\rangle\right)$ Eigenvalue $-J$

$\frac{1}{\sqrt{6}}\left(|{-}{-}{+}\rangle - 2|{-}{+}{-}\rangle + |{+}{-}{-}\rangle\right)$

8.

$\boxed{4}$ We rewrite $\hat{H}$ as

$$\hat{H} = \frac{J}{2\hbar^2}\left[(S_1+S_2+S_3+S_4)^2 - (S_1+S_3)^2 - (S_2+S_4)^2\right]$$
$$+ \frac{J'}{2\hbar^2}\left[(S_1+S_3)^2 + (S_2+S_4)^2 - S_1^2 - S_2^2 - S_3^2 - S_4^2\right]$$



To solve problem rapidly (you could also construct the 16x16 matrix and diagonalize it!) you use the rules for adding angular momentum, eg

$1/2 + 1/2 = 0, 1$ and $1 + 1 = 0, 1, 2$.

You also recall that J^2 gives $j(j+1)\hbar^2$ when acting on $|j\, m\rangle$.

4/ cont'd

So we look at the different cases and compute E

$S_1 + S_3$	$S_2 + S_4$	$S_1+S_2+S_3+S_4$	E
0	0	0	$-3J'/2$
0	1	1	$-J'/2$
1	0	1	$-J'/2$
1	1	0	$-2J + J'/2$
		1	$-J + J'/2$
		2	$+J + J'/2$

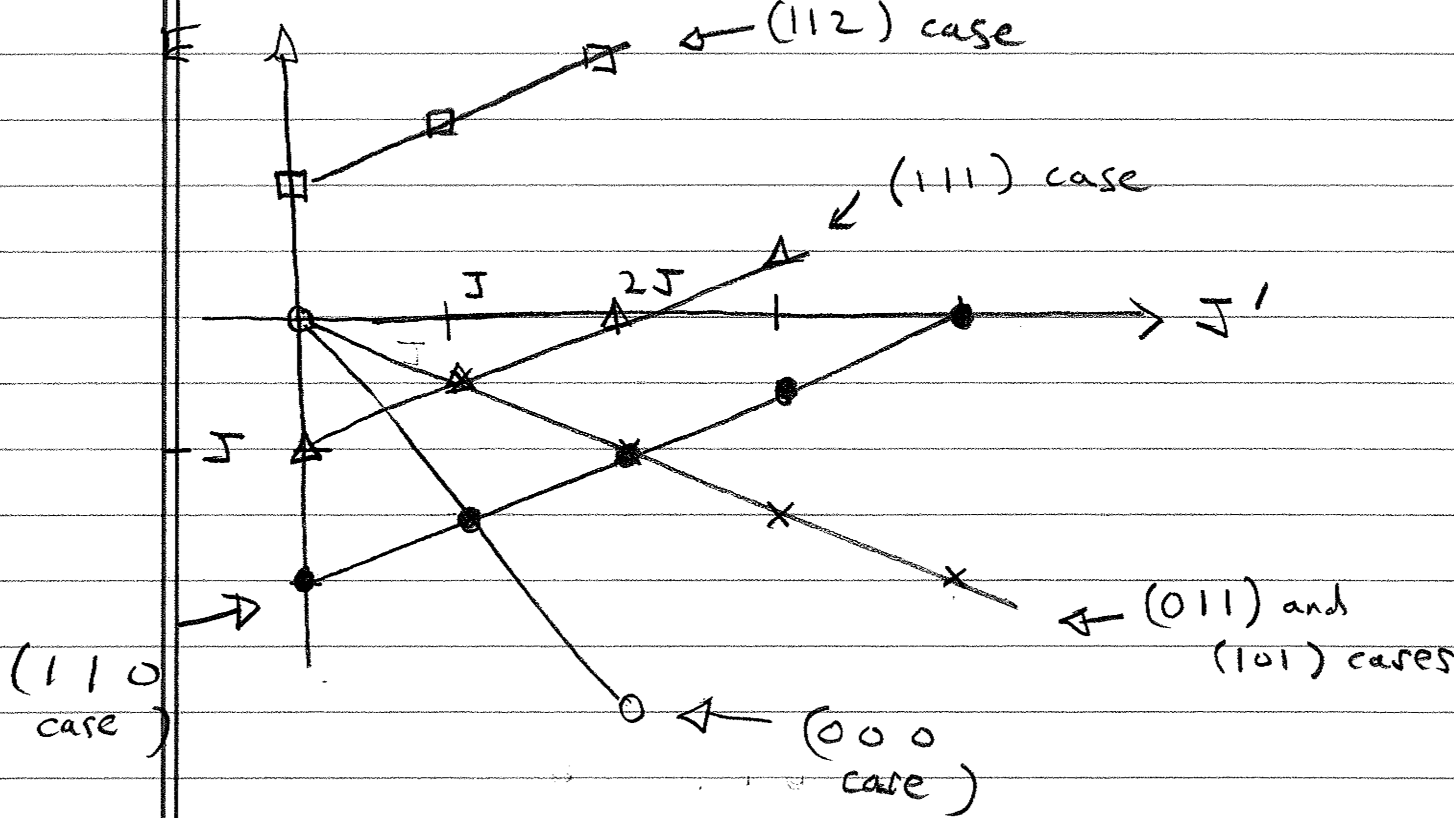
A typical calculation is like this (for $S_1+S_3=1$; $S_2+S_4=0$)

$$H = \frac{J}{2\hbar^2}\Big[\underbrace{(S_1+S_2+S_3+S_4)^2}_{2\hbar^2} \underbrace{-(S_1+S_3)^2}_{-2\hbar^2} \underbrace{-(S_2+S_4)^2}_{\phi}\Big]$$

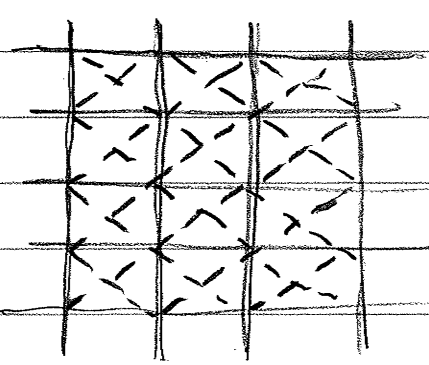
$$+ \frac{J'}{2\hbar^2}\Big[\underbrace{(S_1+S_3)^2}_{2\hbar^2} + \underbrace{(S_2+S_4)^2}_{\phi} \underbrace{-S_1^2 - S_2^2 - S_3^2 - S_4^2}_{-4(\frac{3}{4}\hbar^2)}\Big]$$

↓ Material from now on we will discuss in class (ie not expected on your HW sol'ns)

4/cont'd. Let's plot these energies



What you can see is that the lowest energy is e(110) for $J' < J$ but switches to (000) at $J' > J$. This "level crossing" becomes an actual phase transition if one studies a big lattice:



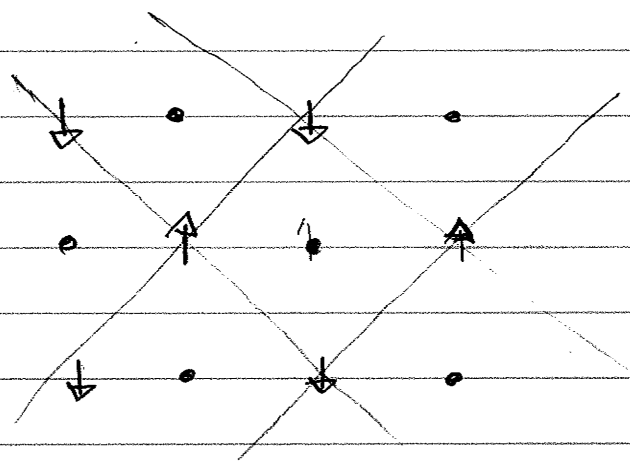
of J bonds and J' bonds

4 cont'd. More specifically for $J' \ll J$ the ground state

$$\begin{array}{cccc} \uparrow & \downarrow & \uparrow & \downarrow \\ \downarrow & \uparrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \uparrow & \downarrow \end{array}$$

is an antiferromagnet with neighboring spins along sides of square antiparallel

But if J' dominates one has two interpenetrating antiferromagnets with spins along diagonals antiparallel



one of the AF

(the other has spins at centers of squares

Amazingly the physics of the model for intermediate $J' \approx J$ is <u>not</u> resolved!!

$\boxed{5}$ Each of the N spins has 2 choices $S_\ell^z = \pm 1/2$ so the Hilbert space has dimension 2^N. (We did $N=2$ which had a $2^2=4$ dim space and $N=4$ which had a $2^4=16$ dim space.)

The basic idea of writing a code to generate the matrix is this: ① You use a bunch of loops to allow the different values of $\{S_\ell^z\}$ ② For each $\{S_\ell^z\}$ you compute the associated basis number n. ③ You then act with $\hat{H}$ by looking for neighboring $(+-)$ pairs of S^z and you exchange them $(-+)$ you compute the new basis number m with that exchange and then set $H_{nm} = J/2$. ④ To get the diagonal entries H_{nn} just compute $J\sum_{\langle ij\rangle} S_i^z S_j^z$.

SEE PAGE 15 FOR SOME MORE DETAILS

5 cont'd

How big N is doable

<u>CPU consideration</u> To diagonalize a matrix of dimension D takes D^3 operations. A computer can do $\sim 10^9$ operations per second (GHz chip), so to diagonalize the matrix for an N site spin 1/2 Heisenberg model will take cpu time (in seconds)

$$T \sim (2^N)^3 / 10^9$$

N	T (sec)
5	$2^{15}/10^9 \sim 3 \cdot 10^{-5}$ sec
10	$2^{30}/10^9 \sim 1$ sec
15	$2^{45}/10^9 \sim 3 \cdot 10^4$ sec $\sim 1/2$ day
20	$2^{60}/10^9 \sim 10^9$ sec ~ 30 years

So limit is ~ 15 sites from CPU considerations

20 sites takes 30 years!

5 cont'd

Memory consideration

To store a # in a computer you use 64 bits = 8 bytes

so to store a matrix of dimension D takes D^2

numbers (ignoring sparsity) or $8D^2$ bytes.

A typical computer memory might be 16 Gbytes

so to store the matrix $8D^2 < 16 \cdot 10^9$

$$D < \sqrt{2 \cdot 10^9} \sim 50000$$

recall $D = 2^N$ so this puts a limit

$$2^N = 50000$$

$$N = \ln(5000)/\ln 2 = 16$$

limit from memory considerations

Interesting that CPU and memory limit roughly the same!

Discuss in class (perhaps). Not expecting you to know this:

A specific way to assign a basis # to a basis vector $|S_1^z\ S_2^z\ S_3^z \ldots S_N^z\rangle$ is

$$n = S_1^z + 1/2 + 2(S_2^z + 1/2) + 4(S_3^z + 1/2) + 8(S_4^z + 1/2) + \ldots$$

Thus, for example, for $N = 4$ sites

$|- - - -\rangle \longrightarrow n = 0$

$|+ - - -\rangle \longrightarrow n = 1$

$|- + - -\rangle \longrightarrow n = 2$

$|+ + - -\rangle \longrightarrow n = 3$

$\vdots$

$|+ + + +\rangle \longrightarrow n = 15$

To "undo" this and get $|s_1^z\, s_2^z\, s_3^z \ldots\rangle$ from n

notice the following eqns work!

$$s_1^z + 1/2 = \mathrm{mod}(n, 2)$$

This works because all the $s_2^z + 1/2$, $s_3^z + 1/2$, ...

are multiplied by powers of 2 so give ϕ when

you take mod(, 2)!

Once you get s_1^z the trick to get s_2^z is

$$\frac{1}{2}\left[n - (s_1^z + 1/2)\right] = s_2^z + 1/2 + 2(s_3^z + 1/2) + \ldots$$

so $$s_2^z - 1/2 = \mathrm{mod}\left(\frac{1}{2}\left[n - (s_1^z + 1/2)\right], 2\right)$$

by same reasoning that gave us $s_1^z + 1/2$.

So we have exhibited a way to

assign a basis number n to each $|s_1^z\, s_2^z \ldots s_N^z\rangle$

and also given the basis number how to get $\{s_e^z\}$.