If $p < q$, as assumed, there is no solution and the integral vanishes. The same result obviously must follow if $p > q$.

For the remaining case, $p = q$, we may still have the single term corresponding to $i = q - m$. Putting Eq. 12.97 into Eq. 12.96, we have

$$\int_{-1}^{1} [P_q^m(x)]^2\, dx = \frac{(-1)^{q+2m}(q+m)!}{2^{2q} q!\, q!\, (2m)!\, (q-m)!} \int_{-1}^{1} X^q \left(\frac{d^{2m}}{dx^{2m}} X^m\right)\left(\frac{d^{2q}}{dx^{2q}} X^q\right) dx. \quad (12.101)$$

Since

$$X^m = (x^2 - 1)^m = x^{2m} - m x^{2m-2} + \cdots, \quad (12.102)$$

$$\frac{d^{2m}}{dx^{2m}} X^m = (2m)!, \quad (12.103)$$

Eq. 12.101 reduces to

$$\int_{-1}^{1} [P_q^m(x)]^2\, dx = \frac{(-1)^{q+2m}(2q)!\,(q+m)!}{2^{2q} q!\, q!\, (q-m)!} \int_{-1}^{1} X^q\, dx. \quad (12.104)$$

The integral on the right is just

$$(-1)^q \int_0^{\pi} \sin^{2q+1}\theta\, d\theta = \frac{(-1)^q 2^{2q+1} q!\, q!}{(2q+1)!} \quad (12.105)$$

(cf. Exercise 10.4.9). Combining Eqs. 12.104 and 12.105, we have the orthogonality integral

$$\int_{-1}^{1} P_p^m(x)\, P_q^m(x)\, dx = \frac{2}{2q+1} \cdot \frac{(q+m)!}{(q-m)!}\, \delta_{p,q} \quad (12.106)$$

or, in spherical polar coordinates,

$$\int_0^{\pi} P_p^m(\cos\theta)\, P_q^m(\cos\theta)\sin\theta\, d\theta = \frac{2}{2q+1} \cdot \frac{(q+m)!}{(q-m)!}\, \delta_{p,q}. \quad (12.107)$$

The orthogonality of the Legendre polynomials is actually a special case of this result, obtained by setting $m$ equal to zero; that is, for $m = 0$, Eq. 12.106 reduces to Eqs. 12.43 and 12.48.

It is possible to develop an orthogonality relation for associated Legendre functions of the same lower index but different upper index. We find

$$\int_{-1}^{1} P_n^m(x) P_n^k(x)(1 - x^2)^{-1}\, dx = \frac{(n+m)!}{m(n-m)!}\, \delta_{m,k}. \quad (12.108)$$

Note that a new weighting factor, $(1 - x^2)^{-1}$, has been introduced. This form is essentially a mathematical curiosity. In physical problems orthogonality of the $\varphi$ dependence ties the two upper indices together and leads to Eq. 12.107.

EXAMPLE 12.5.1. MAGNETIC INDUCTION FIELD OF A CURRENT LOOP.

Like the other differential equations of mathematical physics, the associated

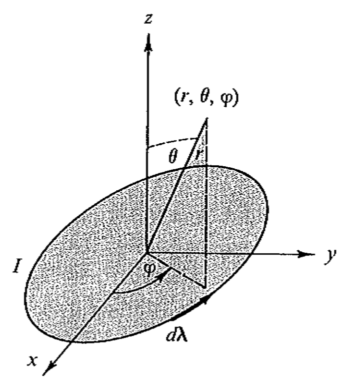


FIG. 12.12 Circular current loop

Legendre equation is likely to pop up quite unexpectedly. As an illustration, consider the magnetic induction field **B** and magnetic vector potential **A** created by a single circular current loop in the equatorial plane (Fig. 12.12).

We know from electromagnetic theory that the contribution of current element $I\, d\boldsymbol{\lambda}$ to the magnetic vector potential is

$$d\mathbf{A} = \frac{\mu_0}{4\pi} \frac{I\, d\boldsymbol{\lambda}}{r}. \tag{12.109}$$

This, plus the symmetry of our system, shows that **A** has only a $\boldsymbol{\varphi}_0$-component and that the component is independent of $\varphi$[1]

$$\mathbf{A} = \boldsymbol{\varphi}_0\, A_\varphi(r, \theta). \tag{12.110}$$

By Maxwell's equations

$$\nabla \times \mathbf{H} = \mathbf{J}, \qquad (\partial \mathbf{D}/\partial t = 0, \text{ mks units}). \tag{12.111}$$

Since

$$\mu_0 \mathbf{H} = \mathbf{B} = \nabla \times \mathbf{A}, \tag{12.112}$$

we have

$$\nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J}, \tag{12.113}$$

where **J** is the current density. In our problem **J** is zero everywhere except in the current loop. Therefore, away from the loop,

$$\nabla \times \nabla \times \boldsymbol{\varphi}_0\, A_\varphi(r, \theta) = 0, \tag{12.114}$$

which introduces Eq. 12.110.

Using the expression for the curl in spherical polar coordinates (Section 2.4), we obtain (Example 2.4.2)

$$\nabla \times \nabla \times \boldsymbol{\varphi}_0\, A_\varphi(r, \theta) = \boldsymbol{\varphi}_0 \left[ -\frac{\partial^2 A_\varphi}{\partial r^2} - \frac{2}{r}\frac{\partial A_\varphi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 A_\varphi}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial}{\partial \theta}(\cot \theta A_\varphi) \right]$$

$$= 0. \tag{12.115}$$

[1] Pair off corresponding current elements $I\, d\boldsymbol{\lambda}(\varphi_1)$ and $I\, d\boldsymbol{\lambda}(\varphi_2)$, where $\varphi - \varphi_1 = \varphi_2 - \varphi$.

Letting $A_\varphi(r, \theta) = R(r)\Theta(\theta)$ and separating variables, we have

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0, \tag{12.116}$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} + n(n+1)\Theta - \frac{\Theta}{\sin^2\theta} = 0. \tag{12.117}$$

The second equation is the associated Legendre equation (12.83) with $m = 1$, and we may immediately write

$$\Theta(\theta) = P_n^1(\cos\theta). \tag{12.118}$$

The separation constant $n(n+1)$ was chosen to keep this solution well behaved.

By trial, letting $R(r) = r^\alpha$, we find that $\alpha = n, -n-1$. The first possibility is discarded, for our solution must vanish as $r \to \infty$. Hence

$$A_{\varphi n} = \frac{b_n}{r^{n+1}} P_n^1(\cos\theta) = c_n\left(\frac{a}{r}\right)^{n+1} P_n^1(\cos\theta) \tag{12.119}$$

and

$$A_\varphi(r, \theta) = \sum_{n=1}^{\infty} c_n\left(\frac{a}{r}\right)^{n+1} P_n^1(\cos\theta), \qquad (r > a). \tag{12.120}$$

Here $a$ is the radius of the current loop.

Since $A_\varphi$ must be invariant to reflection in the equatorial plane by the symmetry of our problem,

$$A_\varphi(r, \cos\theta) = A_\varphi(r, -\cos\theta), \tag{12.121}$$

the parity property of $P_n^m(\cos\theta)$ (Eq. 12.93) shows that $c_n = 0$ for $n$ even.

To complete the evaluation of the constants, we may use Eq. 12.119 to calculate $B_z$ along the $z$-axis $[B_z = B_r(r, \theta = 0)]$ and compare with the expression obtained from the Biot and Savart law. This is the same technique that is used in Example 12.3.3. We have

$$\begin{aligned} B_r &= \nabla \times \mathbf{A}\,\Big|_r \\ &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, A_\varphi)\right] \\ &= \frac{\cot\theta}{r} A_\varphi + \frac{1}{r}\frac{\partial A_\varphi}{\partial\theta}. \end{aligned} \tag{12.122}$$

Using

$$\begin{aligned} \frac{\partial P_n^1(\cos\theta)}{\partial\theta} &= -\sin\theta\frac{dP_n^1(\cos\theta)}{d(\cos\theta)} \\ &= -\frac{1}{2}P_n^2 + \frac{n(n+1)}{2}P_n^0 \end{aligned} \tag{12.123}$$

and Eq. 12.87 with $m = 1$,

$$P_n^2(\cos\theta) - \frac{2\cos\theta}{\sin\theta} P_n^1(\cos\theta) + n(n+1)P_n(\cos\theta) = 0, \tag{12.124}$$

we obtain

$$B_r(r,\theta)=\sum_{n=1}^{\infty} c_n n(n+1)\frac{a^{n+1}}{r^{n+2}}P_n(\cos\theta), \qquad r>a \tag{12.125}$$

(for all $\theta$). In particular, for $\theta = 0$,

$$B_r(r,0)=\sum_{n=1}^{\infty} c_n n(n+1)\ \frac{a^{n+1}}{r^{n+2}}. \tag{12.126}$$

We may also obtain

$$B_\theta(r,\theta)=-\frac{1}{r}\frac{\partial(rA_\varphi)}{\partial r}$$

$$=\sum_{n=1}^{\infty} c_n n\frac{a^{n+1}}{r^{n+2}}P_n^1(\cos\theta), \qquad r>a. \tag{12.127}$$

The Biot and Savart law states that

$$d\mathbf{B}=\frac{\mu_0}{4\pi}I\frac{d\boldsymbol{\lambda}\times\mathbf{r}_0}{r^2} \qquad \text{(mks units).} \tag{12.128}$$

We now integrate over the perimeter of our loop (radius $a$); the magnetic induction field along the $z$-axis is $\mathbf{k}B_z$, where

$$B_z=\frac{\mu_0 I}{2}a^2(a^2+z^2)^{-3/2}$$

$$=\frac{\mu_0 I}{2}\frac{a^2}{z^3}\left(1+\frac{a^2}{z^2}\right)^{-3/2} \tag{12.129}$$

Expanding by the binomial theorem

$$B_z=\frac{\mu_0 I}{2}\frac{a^2}{z^3}\left[1-\frac{3}{2}\left(\frac{a}{z}\right)^2+\frac{15}{8}\left(\frac{a}{z}\right)^4-\cdots\right]$$

$$=\frac{\mu_0 I}{2}\frac{a^2}{z^3}\sum_{r=0}^{\infty}(-1)^r\frac{(2r+1)!!}{(2r)!!}\left(\frac{a}{z}\right)^{2r}, \qquad z>a. \tag{12.130}$$

Equating Eqs. 12.126 and 12.130 term by term (with $r = z$),[1] we find

$$c_1=\frac{\mu_0 I}{4}, \qquad c_3=-\frac{\mu_0 I}{16}, \qquad c_2=c_4=\cdots=0.$$

$$c_n=(-1)^{(n-1)/2}\frac{\mu_0 I}{2n(n+1)}\times\frac{(n/2)!}{[(n-1)/2]!(\frac{1}{2})!}, \qquad n \text{ odd}. \tag{12.131}$$

Equivalently, we may write

$$c_{2n+1}=(-1)^n\frac{\mu_0 I}{2^{2n+2}}\cdot\frac{(2n)!}{n!(n+1)!}=(-1)^n\frac{\mu_0 I}{2}\cdot\frac{(2n-1)!!}{(2n+2)!!} \tag{12.132}$$

[1] The descending power series is also unique.

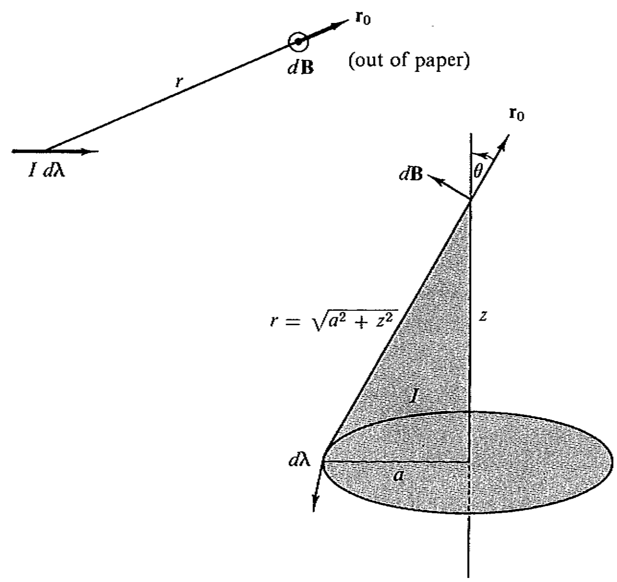


FIG. 12.13 Law of Biot and Savart applied to a circular loop

and

$$A_\varphi(r, \theta) = \left(\frac{a}{r}\right)^2 \sum_{n=0}^{\infty} c_{2n+1} \left(\frac{a}{r}\right)^{2n} P_{2n+1}^1(\cos\theta), \tag{12.133}$$

$$B_r(r, \theta) = \frac{a^2}{r^3} \sum_{n=0}^{\infty} c_{2n+1}(2n+1)(2n+2)\left(\frac{a}{r}\right)^{2n} P_{2n+1}(\cos\theta), \tag{12.134}$$

$$B_\theta(r, \theta) = \frac{a^2}{r^3} \sum_{n=0}^{\infty} c_{2n+1}(2n+1)\left(\frac{a}{r}\right)^{2n} P_{2n+1}^1(\cos\theta). \tag{12.135}$$

These fields may be described in closed form by the use of elliptic integrals. Ex. 5.8.4 is an illustration of this approach. A third possibility is direct integration of Eq. 12.109 by expanding the factor $1/r$ as a Legendre polynomial generating function. The current is specified by Dirac delta functions. These methods have the advantage of yielding the constants $c_n$ directly.

A comparison of magnetic current loop dipole fields and finite electric dipole fields may be of interest. For the magnetic current loop dipole the preceding analysis gives

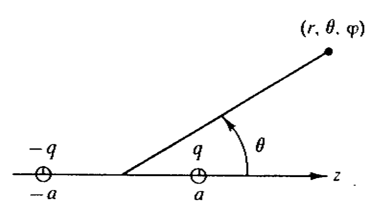


FIG. 12.14

loop

(12.133)

(cos $\theta$), (12.134)

(12.135)

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$$B_r(r, \theta) = \frac{\mu_0 I}{2} \frac{a^2}{r^3}\left[P_1 - \frac{3}{2}\left(\frac{a}{r}\right)^2 P_3 + \cdots\right], \tag{12.136}$$

$$B_\theta(r, \theta) = \frac{\mu_0 I}{4} \frac{a^2}{r^3}\left[P_1^1 - \frac{3}{4}\left(\frac{a}{r}\right)^2 P_3^1 + \cdots\right]. \tag{12.137}$$

From the finite electric dipole potential of Section 12.1 we have

$$E_r(r, \theta) = \frac{qa}{\pi\varepsilon_0 r^3}\left[P_1 + 2P_3\left(\frac{a}{r}\right)^2 + \cdots\right], \tag{12.138}$$

$$E_\theta(r, \theta) = \frac{qa}{2\pi\varepsilon_0 r^3}\left[P_1^1 + \left(\frac{a}{r}\right)^2 P_3^1 + \cdots\right]. \tag{12.139}$$

The two fields agree in form as far as the leading term is concerned, and this is the basis for calling them both dipole fields.

As with electric multipoles, it is sometimes convenient to talk about *point* magnetic multipoles. For the dipole case, Eqs. 12.136 and 12.137, the point dipole is formed by taking the limit $a \to 0$, $I \to \infty$ with $Ia^2$ held constant. With $\mathbf{n}$ a unit vector normal to the current loop (positive sense by right hand rule, Section 1.10) the magnetic moment $\mathbf{m}$ is given by $\mathbf{m} = \mathbf{n} I \pi a^2$.

## EXERCISES

**12.5.1** Prove that

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x),$$

where $P_n^m(x)$ is defined by

$$P_n^m(x) = \frac{1}{2^n\, n!}(1-x^2)^{m/2} \frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n.$$

*Hint*. One approach is to apply Leibnitz' formula to $(x+1)^n(x-1)^n$.

**12.5.2** Show that

$$P_{2n}^1(0) = 0,$$

$$P_{2n+1}^1(0) = (-1)^n \frac{(2n+1)!}{(2^n n!)^2} = (-1)^n \frac{(2n+1)!!}{(2n)!!},$$

by each of the three methods:
(a) use of recurrence relations,
(b) expansion of the generating function, and
(c) Rodrigues' formula.