①

System with energy levels $E_1, E_2, \dots E_N$

Boltzmann $\quad p_n = e^{-E_1/k_BT} \Big/ \underbrace{\sum_{n=1}^{N} e^{-E_n/k_BT}}_{Z} \quad \beta = 1/k_BT$

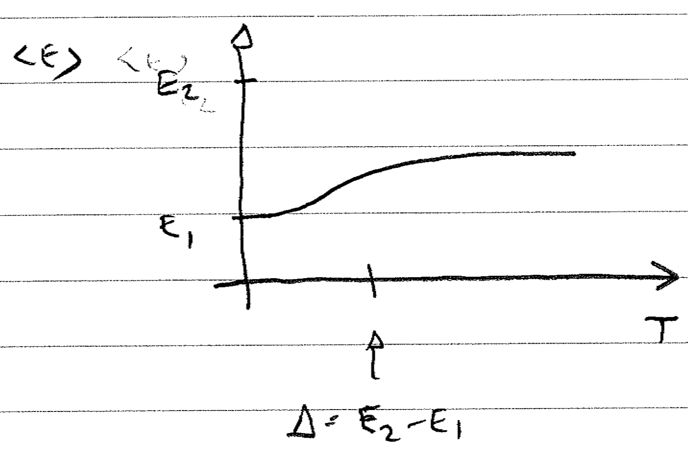
"Canonical Ensemble"

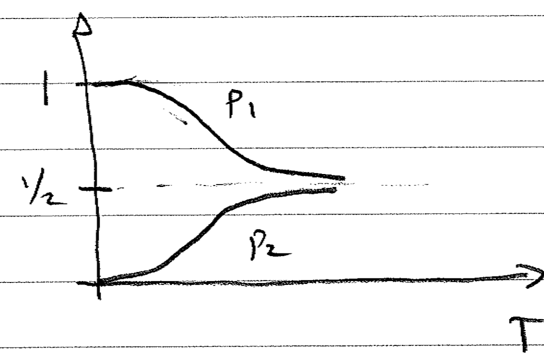
Once p_n known can compute averages

$$\langle A \rangle = \sum A_n p_n$$

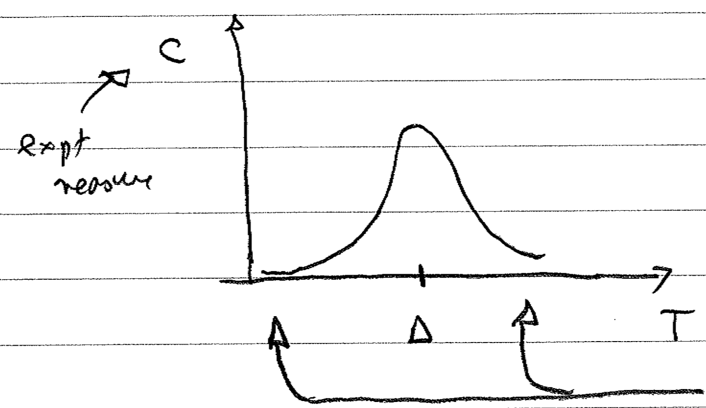
$$\langle E \rangle = \sum E_n p_n = \sum E_n e^{-\beta E_n} \Big/ \sum e^{-\beta E_n}$$

Examples $\quad N = 2 \quad E_1, E_2$





$C = d\langle E \rangle / dT$ ◁— define in words



physics: Response function has maxima at Energy level differences

Comment on limits (after doing Ideal gas example)

Focus on computation of Z.

why?

$$-\frac{\partial}{\partial\beta}\ln Z = \frac{1}{Z}\left(-\frac{\partial}{\partial\beta}\right)\sum_n e^{-\beta E_n} = \frac{1}{Z}\sum E_n e^{-\beta E_n} = \langle E\rangle$$

$E_n \to E(p) = \vec{p}^2/2m$ continuum

$$Z(\beta) = \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z \; e^{-\beta(p_x^2+p_y^2+p_z^2)/2m}$$

$$= \left(\int_{-\infty}^{\infty} dp \; e^{-\beta p^2/2m}\right)^3 \qquad \int e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$$

$$= \left(\sqrt{2\pi m k_B T}\right)^3 = \left(2\pi m/\beta\right)^{3/2}$$

$$\ln Z(\beta) = 3/2 \ln 2\pi m - 3/2 \ln\beta$$

$$-\frac{\partial}{\partial\beta}\ln Z = +3/2 \; 1/\beta = 3/2 \, k_B T \qquad \text{Recognize}$$

$H = \sum_i \vec{p}_i^{\,2}/2m$ many particles

$$Z(\beta) = \left(2\pi m/\beta\right)^{3N/2}$$

$$-\partial/\partial\beta \ln Z = 3/2 N k_B T$$

If you want to keep track of $\vec{x}$ also Box volume V

$$Z(\beta) = V^N\left(\frac{2\pi m}{\beta}\right)^{3/2} \qquad \text{from } \int d\vec{r}_i$$

?✗ $$p = \frac{+1}{\beta}\frac{\partial}{\partial V}\ln Z = \frac{1}{\beta}\frac{N}{V} \qquad pV = Nk_B T$$

2A

Impt principle

$$E = E_A + E_B$$

$$Z = \sum_A \sum_B e^{-\beta(E_A + E_B)} = Z_A Z_B$$

$$\langle E \rangle = -\frac{\partial}{\partial\beta} \ln Z = -\frac{\partial}{\partial\beta} \ln Z_A - \frac{\partial}{\partial\beta} \ln Z_B$$

$$= \langle E_A \rangle + \langle E_B \rangle$$

This is why noninteracting problems so easy to solve

"divide and conquer"

Many Body Problem
Σ

Quantum Stat Mech $\quad E_n, |\phi_n\rangle \ldots$ same ideas!

(3)

$$E = -\vec{\mu}\cdot\vec{B}$$ magnetic dipole in $\vec{B}$ field

B in z direction $\rightarrow$ $\mu = \frac{e}{2mc}\vec{L}$ $\rightarrow$ reverse

Two level system $\quad E = -B\,\mu_z$

spin $1/2$

$$= -B\,\frac{e\hbar}{2mc}\left(\pm \tfrac{1}{2}\right)$$

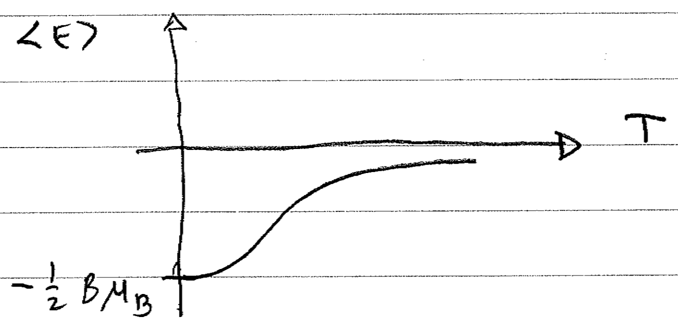
$\leftarrow \mu_B$

$c = \frac{1}{2}(e^{x} + e^{-x})$

$$Z = e^{\beta B \mu_B/2} + e^{-\beta B \mu_B/2} = 2\cosh\tfrac{1}{2}(\beta B \mu_B)$$

$$\langle E\rangle = -\partial/\partial\beta \ln Z = -\frac{1}{\cosh\frac{1}{2}\beta B\mu_B}\left(+\sinh \tfrac{1}{2}\beta B\mu_B\right)\tfrac{1}{2}B\mu_B$$

$$= -\tfrac{1}{2}B\mu_B \tanh \tfrac{1}{2}\beta B\mu_B$$



Better: Use.

$$E = -B\,g\,\mu_B\left(\pm \tfrac{1}{2}\right)$$

g = gyromagnetic ratio

$$\mu = g\,\frac{\mu_B}{\hbar}\,S$$

$$\mu_B = \frac{e\hbar}{mc}$$

Other Expectation values

Magnetic moment

$$\langle \mu\rangle = \sum \mu_n p_n = Z^{-1}\left(\frac{\mu_B}{2}e^{+\beta B\mu_B/2} - \frac{\mu_B}{2}e^{-\beta B\mu_B/2}\right)$$

$$= \frac{\mu_B}{2}\tanh\tfrac{1}{2}\beta B\mu_B$$

Magnetic susceptibility

$$\chi = \frac{d\langle\mu\rangle}{dB} = \frac{\mu_B}{2}\left(\mathrm{sech}^2\tfrac{1}{2}\beta B\mu_B\right)\tfrac{1}{2}\beta\mu_B$$

$$= \frac{\mu_B^2}{4}\,\underbrace{\beta}_{\frac{1}{T}}\,\underbrace{\mathrm{sech}^2\tfrac{1}{2}\beta B\mu_B}_{1 \text{ at low } \beta}$$ curie law

Nice MMP in Stat Mech arises from problems where degrees of freedom interact.

$$E = -B\mu_B \tfrac{1}{2}(S_1 + S_2 + \dots S_N)$$

$\uparrow$ ± 1 $\qquad Z = (3)^N$

simplest interaction

$$\underset{S_1}{\circ}\!-\!\!-\underset{S_2}{\circ}\!-\!\!-\underset{S_3}{\circ}\!-\!\!-\underset{S_4}{\circ}\ \dots$$

$$E = -J\,(S_1 S_2 + S_2 S_3 + S_3 S_4 + \dots S_N S_1)$$

lowest $E = -JN$ F

highest $E = +JN$ AF

Compute Z?

$$Z = \sum_{S_1=\pm 1}\sum_{S_2=\pm 1}\dots\sum_{S_N=\pm 1} e^{\beta J S_1 S_2}\, e^{\beta J S_2 S_3}\dots$$

$M(S_1, S_2)$ matrix with M.E.

$$= \begin{pmatrix} e^{+\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{+\beta J} \end{pmatrix}$$

$$Z = \sum_{S_1}\sum_{S_2}\dots\sum_{S_N} \underbrace{M(S_1 S_2)\, M(S_2 S_3)}_{M^2(S_1, S_3)}\dots M(S_N S_1)$$

(I.)

$$Z = \sum_{s_1} m^N(s_1\, s_1) = \mathrm{Tr}\, M^N = \lambda_1^N + \lambda_2^N$$

$$\cancel{\lambda =} \quad (e^{\beta J} - \lambda)^2 = e^{-2\beta J}$$

$$\lambda = 2\cosh\beta J + 2\sinh\beta J$$

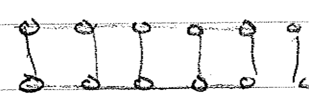
$$Z = (2\cosh\beta J)^N + (2\sinh\beta J)^N$$

~~What about 2 d?~~ $\quad N \to \infty \quad Z \approx (2\cosh\beta J)^N$

Do XY model...

$N = L \times L$

$d = 2$. Matrix of dim 2^L



$$\sum_{\substack{S \\ \text{row } 1}} \sum_{\substack{S \\ \text{row } 2}} \sum_{\substack{S \\ \text{row } 3}} \cdots \sum_{\substack{S \\ \text{row } L}} M(S_{\text{row } 1}, S_{\text{row } 2})\, M(S_{\text{row } 2}, S_{\text{row } 3}) \cdots$$

Onsager story ...