Spherical Bessel Functions

We solved Laplaces eqn $\nabla^2\psi = 0$ in

spherical coordinates:

$$\psi(r,\theta,\phi) = \sum\left[a_{\ell m} r^{\ell} + b_{\ell m} r^{-(\ell+1)}\right] Y_{\ell m}(\theta,\phi)$$

$P_\ell^m(\theta)\, e^{im\phi}$ times some constants

$P_\ell^0(\theta)$ soln of Legendre eqn

$$\left[(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} - \ell(\ell+1)\right] P_\ell^0(x) = 0$$

$$x = \cos\theta$$

$$P_\ell^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m} P_\ell^0(x)$$

What if we have Helmholtz eqn $\nabla^2\psi = -k^2\psi$?

The exact same separation of variables approach yields an identical angular structure $Y_{\ell m}(\theta,\phi)$ but a modified radial eqn

Laplace: $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{\ell(\ell+1)}{r^2}R = 0$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \ell(\ell+1)R = 0$$

Helmholtz $$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2r^2R - \ell(\ell+1)R = 0$$

This can be written

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \left[k^2r^2 - \ell(\ell+1)\right]R = 0$$

If we write $R(kr) = Z(kr)/(kr)^{1/2}$

Then we get (* SB-2A)

$$r^2\frac{d^2Z}{dr^2} + r\frac{dZ}{dr} + \left[k^2r^2 - (\ell+\tfrac{1}{2})^2\right]Z = 0$$

Compare this to Bessel's Eqn

$$x^2\frac{d^2}{dx^2}J_n + x\frac{dJ_n}{dx} + (k^2x^2 - n^2)J_n = 0$$

Looks just like this except n is not an integer!

In fact, we can define $J_\nu(x)$ for any $\nu > -1$

as a soln of

$$\left[x^2\frac{d^2}{dx^2}J_\nu + x\frac{dJ_\nu}{dx} + (k^2x^2 - \nu^2)\right]J_\nu = 0$$

and much of the old machinery is still valid, e.g. $n \to \nu$

$$J_\nu(x) = \sum_{s=0}^{\infty}(-1)^s\frac{1}{s!(s+\nu)!}\left(\frac{x}{2}\right)^{\nu+2s}$$

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x)$$

Just no generating function (required integer n)

SB-2A

$$\frac{dR}{dr} = \frac{d}{dr}\left(\frac{Z}{r^{1/2}}\right) = \frac{1}{r^{1/2}}\frac{dZ}{dr} - \frac{1}{2}\frac{1}{r^{3/2}}Z$$

$$\frac{d^2R}{dr^2} = \frac{1}{r^{1/2}}\frac{d^2Z}{dr^2} - \frac{1}{r^{3/2}}\frac{dZ}{dr} + \frac{3}{4}\frac{1}{r^{5/2}}Z$$

$$r^2\frac{d^2R}{dr^2} = r^{3/2}\frac{d^2Z}{dr^2} - r^{1/2}\frac{dZ}{dr} + \frac{3}{4}\frac{1}{r^{1/2}}Z$$

$$+ \quad 2r\frac{dR}{dr} = \qquad 2r^{1/2}\frac{dZ}{dr} - \frac{1}{r^{1/2}}Z$$

$$r^2\frac{dR}{dr} + 2r\frac{dR}{dr} = r^{3/2}\frac{d^2Z}{dr^2} + r^{1/2}\frac{dZ}{dr} - \frac{1}{4}\frac{1}{r^{1/2}}Z$$

$$r^{3/2}\frac{d^2Z}{dr^2} + r^{1/2}\frac{dZ}{dr} - \frac{1}{4}\frac{1}{r^{1/2}}Z + \frac{1}{r^{1/2}}\left[k^2r^2 - l(l+1)\right]Z = 0$$

$\times$ by $r^{1/2}$

$$r^2\frac{d^2Z}{dr^2} + r\frac{dZ}{dr} + \left[k^2r^2 - \underbrace{l(l+1) - 1/4}_{}\right]Z = 0$$

$$\underbrace{-l^2 - l - 1/4}_{-(l+1/2)^2}$$

SB-3

Because 1/2 integer $\nu = n + 1/2$ arise so often (when combined with $1/\sqrt{x}$) (eg Helmholtz) they are given special name: the

" spherical Bessel functions "

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}(x)$$

little j

$$j_0(x) = \sqrt{\frac{\pi}{2x}} \sum_{s=0}^{\infty} (-1)^s \frac{(x/2)^{2s+\frac{1}{2}}}{s!\,(s+\frac{1}{2})!}$$

$$= \frac{\sin x}{x}$$

(* SB-3ABC

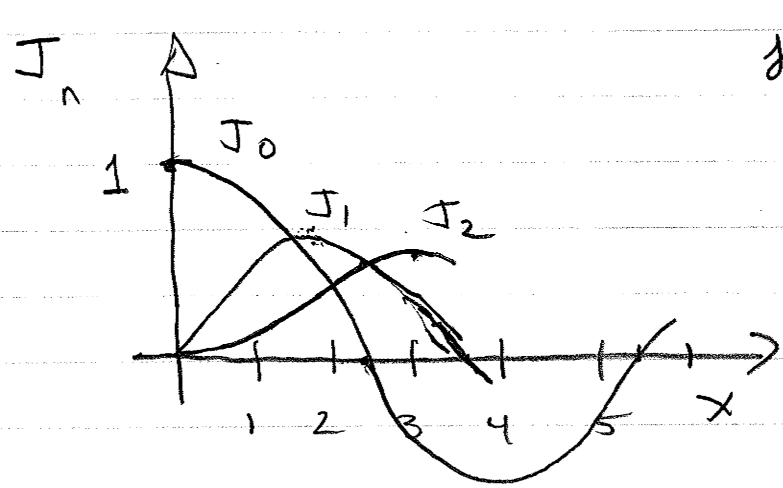
For large x $\quad j_n(x) \to \frac{1}{x}\sin\left(x - \frac{n\pi}{2}\right)$ ← Nontrivial to show!

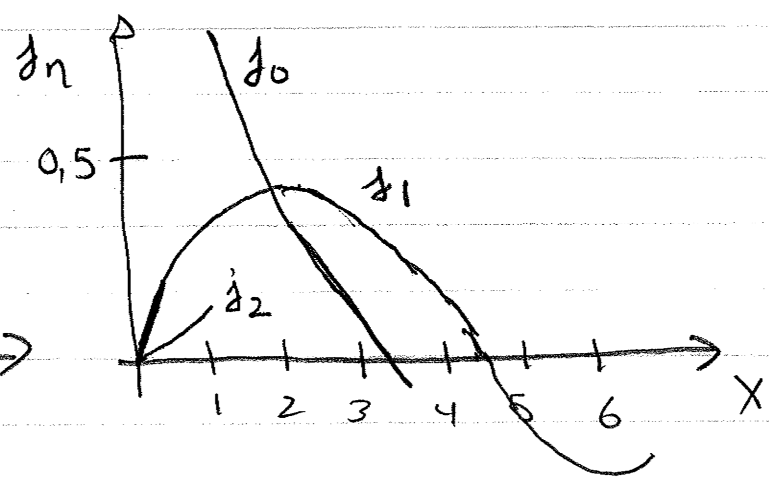
$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$ ← follows from recursion relns

($j_1(x)$ well behaved as $x \to 0$

$$\approx \frac{x - x^3/6 + x^5/120}{x^2} - \frac{1 - x^2/2 + x^4/24}{x}$$

$$= \frac{1}{x} - \frac{x}{6} + \frac{x^3}{120} - \frac{1}{x} + \frac{x}{2} - \frac{x^3}{24} \sim \frac{x}{3} - \frac{x^3}{30} \ldots\ )$$





SB-3A

GAMMA FUNCTION

What is $(5+1/2)!$ or, more generally,

the factorial of a non-integer?!

$$n! = n(n-1)(n-2)\ldots 1$$

We encountered the integral (in HW problems)

$$\int_0^\infty e^{-t} t^{n-1}\, dt = (n-1)!$$ (proof by integration-by parts)

Define the $\underline{\underline{\Gamma}}$ $\underline{\underline{\text{function}}}$

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$$

$z=n$ $\quad \Gamma(n+1) = n!$

but $\Gamma(z)$ defined more generally

Integration by parts still works even if z non integer

$$\Gamma(z+1) = z\,\Gamma(z)$$

<u>Check!</u> $\Gamma(n+1) = n! = n(n-1)! = n\,\Gamma(n)$

SB-3B

$\Gamma(1/2) = ?$ As an example

$$\Gamma(1/2) = \int_0^\infty e^{-t}\, t^{-1/2}\, dt$$

Let $t = x^2$ $\quad dt = 2x\,dx$

$$\Rightarrow \int_0^\infty e^{-x^2} \frac{1}{x}\, 2x\,dx = 2\underbrace{\int_0^\infty e^{-x^2} dx}_{\sqrt{\pi}\,\frac{1}{2}}$$

$$\Gamma(1/2) = \sqrt{\pi}$$

(Many interesting facts eg

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3 \cdots n}{z(z+1)(z+2)\cdots(z+n)}\, n^z$$)

$$\Gamma(3/2) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

5B-3C

Finally, returning to $j_0(x)$

$$j_0(x) = \sqrt{\frac{\pi}{2x}} \sum_{s=0}^{\infty} (-1)^s \frac{(x/2)^{2s+1/2}}{s!\,(s+1/2)!} \longrightarrow \text{gives } \sqrt{\frac{x}{2}}$$

$$= \frac{\sqrt{\pi}}{2} \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{2s} \frac{1}{s!\,(s+1/2)!}$$

$$= \frac{\sqrt{\pi}}{2} \left\{ \frac{1}{0!\,1/2!} - \left(\frac{x}{2}\right)^2 \frac{1}{1!\,(3/2)!} + \left(\frac{x}{2}\right)^4 \frac{1}{2!\,(5/2)!} - \dots \right\}$$

$$= \frac{\sqrt{\pi}}{2} \left\{ \frac{2}{\sqrt{\pi}} - \frac{x^2}{4}\,\frac{4}{3\sqrt{\pi}} + \frac{x^4}{16}\,\frac{1}{2}\,\frac{8}{15\sqrt{\pi}} - \dots \right\}$$

$$= \left\{ 1 - \frac{x^2}{3 \cdot 2} + \frac{x^4}{5 \cdot 4 \cdot 3 \cdot 2} - \dots \right\}$$

$$= \left\{ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right\}$$

$$= \frac{\sin x}{x}$$

$\frac{1}{2}! = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$

$\frac{3}{2}! = \Gamma(5/2) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\sqrt{\pi}$

$\frac{5}{2}! = \Gamma(7/2) = \frac{5}{2}\Gamma(5/2) = \frac{15}{8}\sqrt{\pi}$

SB-4

QM particle in a sphere (radius a)

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$$

$$\Rightarrow \quad r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \Big(\underbrace{\frac{2mE}{\hbar^2}}_{k^2} r^2 - l(l+1)\Big)R = 0$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

for $l=0$ $\quad R = j_0\left(\frac{\sqrt{2mE}}{\hbar} r\right)$

$$\psi(r,\theta,\phi) = A\, j_0\left(\frac{\sqrt{2mE}}{\hbar} r\right) \underbrace{Y_{0,0}(\theta,\phi)}_{\sim \text{ constant}}$$

(Even though j_0 diverges at origin ψ is still integrable there)

$$j_0\left(\frac{\sqrt{2mE}}{\hbar} a\right) = 0$$

$$\frac{\sqrt{2mE_n}}{\hbar} a = x_n$$

roots of j_0 $\qquad$ smallest root of j_0 is π

We proved $j_0(x) = \frac{\sin x}{x}$

So roots of j_0 are $\pi, 3\pi, 5\pi, \ldots$

$$\frac{\sqrt{2mE_n}}{\hbar} = (2n+1)\pi/a \qquad n=0,1,2,\ldots$$

$$2mE_n = (2n+1)^2\pi^2\hbar^2/a^2$$

$$E_n = (2n+1)^2\frac{\pi^2\hbar^2}{2ma^2}$$

$n=0,1,2,\ldots$

So there are a set of levels corresponding to $l=m=0$

Allowing $l=1$ we would get $j_1\left(\frac{\sqrt{2mE}}{\hbar}r\right)$

as radial part and $Y_{1m}(\theta,\phi)$ as angular part.

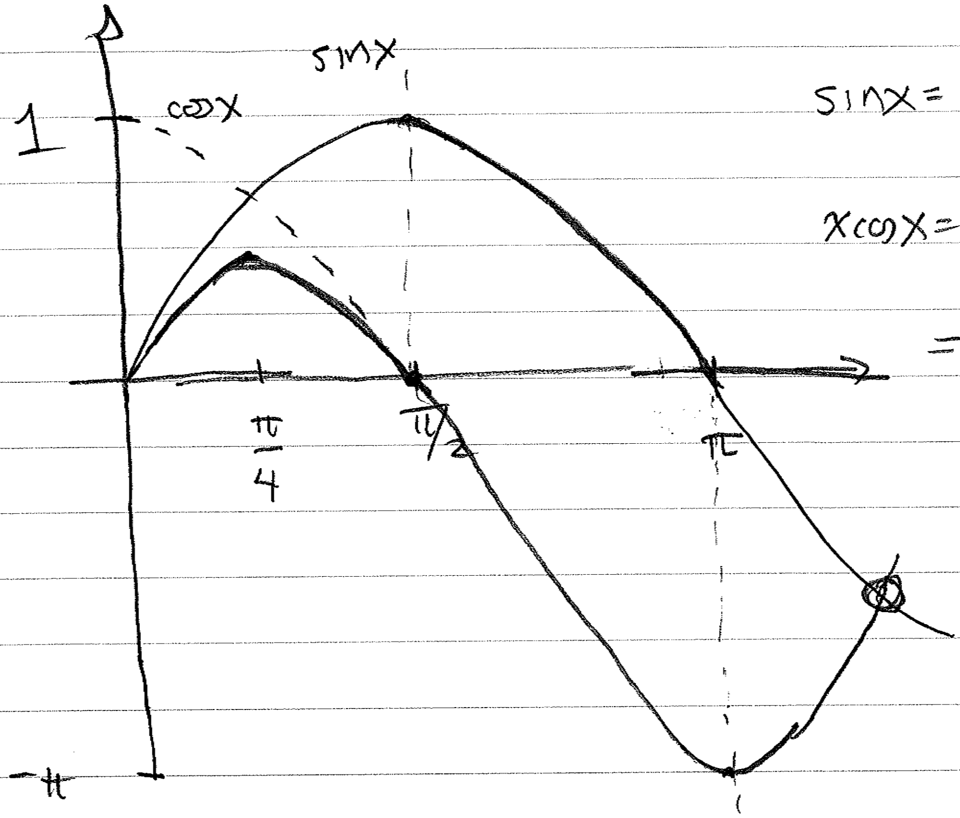
Energy quantization would come from $j_1(x)=0$ roots

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} = 0$$

$$\sin x - x\cos x = 0$$

$$\sin x = x\cos x$$

5B-6



$$\sin x = x - \frac{x^3}{6}$$

$$x \cos x = x\left(1 - \frac{x^2}{2} \cdots\right.$$

$$= x - \frac{x^3}{2} < \sin x$$