Mathematical physics courses typically spend quite a bit of time on "special functions," These are functions which occur very commonly in the solution of the partial differential equations which are central to physics, especially those involving the Laplacian operator, $\nabla^2$

Diffusion Equ

$$D \nabla^2 \psi = \frac{\partial^2 \psi}{\partial t^2}$$

Schroedinger Equ

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V \right] \psi = -i\hbar \frac{\partial \psi}{\partial t}$$

Laplace Equ

$$\nabla^2 \psi = 0$$

Poisson Equ

$$\nabla^2 \psi = 4\pi \rho$$

These special functions often incorporate symmetries of the problem in question.
Before discussing symmetries and special functions in the context of partial differential equations, let's review them in an ordinary differential equation context of classical mechanics. We will see "conservation laws" in the two contexts.

**Central Force Problem**

\[ m \frac{d^2 \mathbf{r}}{dt^2} = \mathbf{F}(\mathbf{r}) = f(r) \mathbf{r} \]

\[ \frac{d}{dt} (\mathbf{r} \times \mathbf{p}) = \mathbf{v} \times \mathbf{p} + \mathbf{r} \times m \frac{d^2 \mathbf{r}}{dt^2} \]

\[ \mathbf{0} \]

(always)

\[ \mathbf{r} \times \mathbf{F} \] (torque)

\[ \neq \text{ for central force} \]

\[ \mathbf{L} = \mathbf{r} \times \mathbf{p} \] is a time independent vector

\[ \mathbf{L} \] is \( \perp \) plane of \( \mathbf{r} \) and \( \mathbf{p} \)

\[ \mathbf{L} \] motion takes place in fixed plane.

\[ \mathbf{L} \] will be connected intimately with "spherical harmonics" \( Y_{lm}(\theta, \phi) \)
For completeness:

Another conserved quantity?

\[ E = \frac{1}{2} m v^2 - \int_{r_0}^r \vec{F}(r') \cdot d\vec{r}' \]

\[
\frac{dE}{dt} = \frac{d}{dt} \left[ \frac{1}{2} m \vec{v} \cdot \vec{v} - \int_{r_0}^r \vec{F}(r') \cdot d\vec{r}' \right]
\]
\[
= m \vec{v} \cdot \vec{a} - \vec{F}(r) \cdot \frac{d\vec{r}}{dt}
\]
\[
= \vec{v} \cdot \{ m \vec{a} - \vec{F} \} = 0
\]

Recall \( \frac{d}{dx} \int_{a}^{x} f(x') dx' = f(x) \)

\[
\frac{d}{dx} \int_{a}^{g(x)} f(x') dx' = f(g(x)) g'(x)
\]

\[ Q1: \text{Under what conditions are orbits "closed"?} \]

\[ Q2: \text{What prevents earth from "falling into sun", i.e., what forbids } r = 0? \]
A1: \( r^2 \) and \( 1/r \) potentials, i.e.

\[ \vec{F}(\vec{r}) = \frac{A \vec{r}}{r^2} \quad \text{and} \quad B \vec{r} \hat{r} \]

give closed orbits. In case of \( 1/r \) "Runge Lenz vector"

A2: Using polar coordinates in plane of motion:

\[ v^2 = \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{\theta}{dt} \right)^2 \]

And also

\[ l = \left| \vec{l} \right| = \left| \vec{r} \times \vec{v} \right| \]

\[ l = \hbar r^2 \frac{d\theta}{dt} \]

So

\[ v^2 = \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{\theta}{m} \frac{l}{r^2} \right)^2 \]

\[ E = \frac{1}{2} m \left( \frac{dr}{dt} \right)^2 + \frac{\ell^2}{2mr^2} + U(r) \]

\[ E \geq -\frac{G M m}{r} \]

Analogy: This is a "one dimensional" energy problem (only \( r \) and \( m, \ell, \theta \))

Sort of like "Radial Schrödinger eqn" left over after

\( \gamma m(\theta, \phi) \) removed from problem.