Mathematical physics courses typically spend quite a bit of time on "special functions." These are functions which occur very commonly in the solution of the partial differential eqns which are central to physics, especially those involving the Laplacian operator, $\nabla^2$

Diffusion Eqn $\quad D\nabla^2\psi = \frac{\partial\psi}{\partial t}$

Schroedinger Eqn $\quad \left[-\frac{\hbar^2}{2m}\nabla^2 + V\right]\psi = -i\hbar\frac{\partial\psi}{\partial t}$

Laplace Eqn $\quad \nabla^2\psi = 0$

Poisson Eqn $\quad \nabla^2\psi = 4\pi\rho$

These special functions often incorporate symmetries of the problem in question.

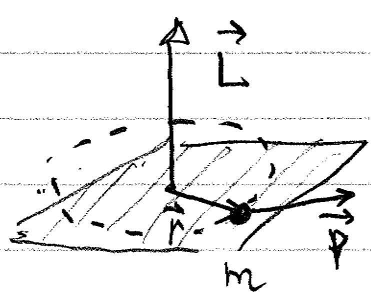
Before discussing symmetries and special functions in the context of partial differential eqns, let's review them in an ordinary differential eqn context of classical mechanics. We will see close connections between "conservation laws" in the two contexts.

## Central Force problem

$$m \, d^2\vec{r}/dt^2 = \vec{F}(\vec{r}) = f(r)\,\hat{r}$$

$$\frac{d}{dt}(\vec{r}\times\vec{p}) = \underbrace{\vec{v}\times\vec{p}}_{\substack{0 \\ \text{(always)}}} + \underbrace{\vec{r}\times m\frac{d^2\vec{r}}{dt^2}}_{\underbrace{\vec{r}\times\vec{F}}_{0 \text{ for central force}} \text{ (torque)}}$$

$\vec{L} = \vec{r}\times\vec{p}$ is a time independent vector




$\vec{L}$ is $\perp$ to plane of $\vec{r}$ and $\vec{p}$

$\Rightarrow$ motion takes place in fixed plane.

$\vec{L}$ will be connected intimately to "spherical harmonics" $Y_{\ell m}(\theta,\phi)$

For completeness:

Another conserved quantity?

The potential energy $u(\vec{r})$

$$E = \frac{1}{2} m v^2 - \int_{\vec{r}_0}^{\vec{r}} \vec{F}(r') \cdot d\vec{r}\,'$$

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2} m \vec{v}\cdot\vec{v} - \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'\right]$$

$$= m\vec{v}\cdot\vec{a} - \vec{F}(\vec{r}) \cdot \frac{d\vec{r}}{dt}$$

$$= \vec{v}\cdot\{m\vec{a} - \vec{F}\} = 0$$

——— o ———

Recall $\displaystyle \frac{d}{dx}\int_a^x f(x')\,dx' = f(x)$

$$\frac{d}{dx}\int_a^{g(x)} f(x')\,dx' = f(g(x))\,g'(x) \quad \text{etc}$$

——— o ———

Q1: Under what conditions are orbits "closed"?

Q2: What prevents earth from "falling into sun", i.e. what forbids $r = 0$?

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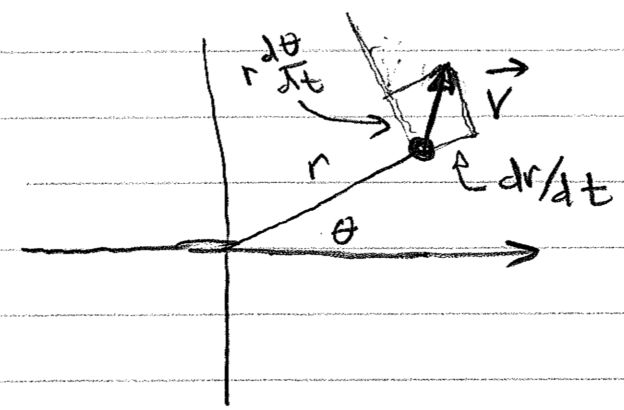
(A1): $r^2$ and $1/r$ potentials, ie

$$\vec{F}(\vec{r}) = \frac{A\hat{r}}{r^2} \quad \text{and} \quad B r \hat{r}$$

give closed orbits. In case of $1/r$ "Runge Lenz vector"

(A2): Using polar coordinates in plane of motion:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$$




and also

$\vec{p}$

$$l = |\vec{L}| = |\vec{r} \times m\vec{v}|$$

$$= mr^2 \frac{d\theta}{dt}$$

So $$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{l}{mr^2}\right)^2$$

$\mathcal{E}_g \quad \frac{-GMm}{r}$

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{l^2}{2mr^2} + \mathcal{U}(r)$$

Analogy: This is a "one dimensional" energy problem (only $r$ and no $\theta$)

sort of like "radial Schroedinger eqn" left over after

$Y_{lm}(\theta,\phi)$ removed from problem.