PHYSICS 204B, WINTER 2011
ASSIGNMENT FOUR

Due Friday, February 11.

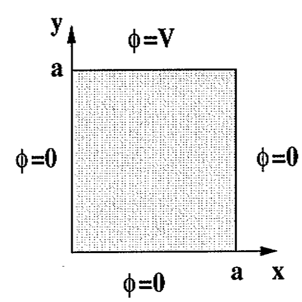
Do Problem 1 and, if you have time, try Problem 2. It should not be too bad a generalization, and, carries the important message that numerical solutions rather easily leap from where analytic solutions are possible to where they are not.

As I mentioned in class, on Friday we can build a code in C for this problem. I will discuss the general theory in room 416 at 2 pm. This will only take 20-30 minutes. Then we will go to room 106 and write the code together. You are of course also welcome to tackle it on your own.

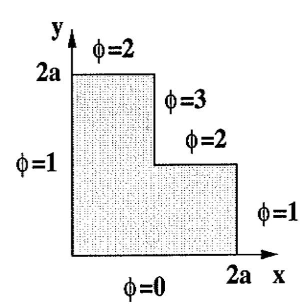
[**1.**] Solve Laplace's equation $\nabla^2\phi = 0$ for the potential $\phi(x, y)$ for the square region in the figure, with the boundary conditions shown with $V = 3$. Use the iterative method discussed in class, whose generalization to $d = 2$ (with $\rho = 0$) is,

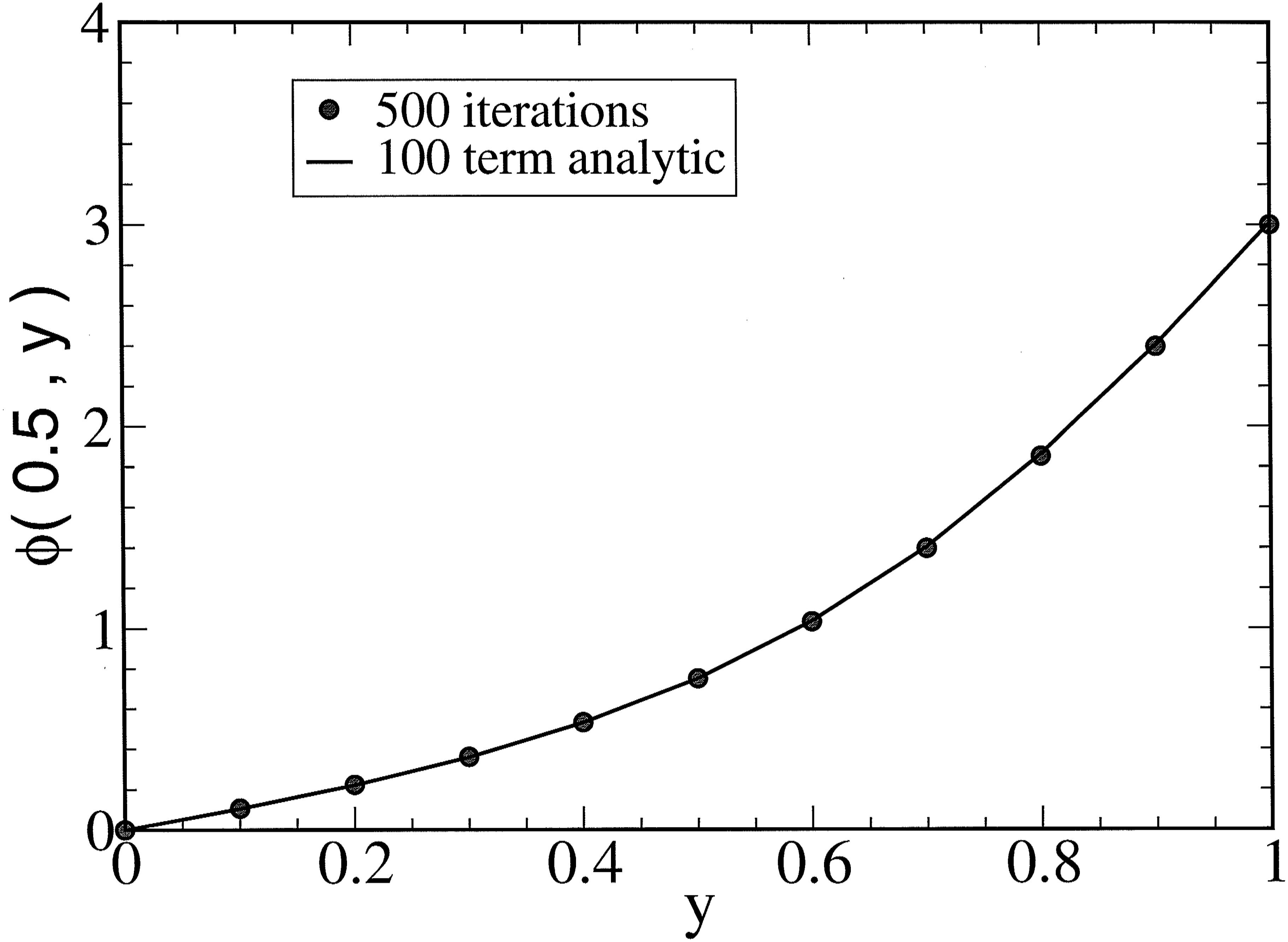
$$\phi(i,j) = \frac{1}{4}[\phi(i+1,j) + \phi(i-1,j) + \phi(i,j-1) + \phi(i,j+1)] \; .$$

Use $dx = 0.1$ and $a = 1$. Compare your solution with the analytic one obtained in class by plotting the two results for $\phi(x, y = a/2)$, $0 < x < a$ on the same graph. Make a second graph comparing the analytic and numeric solutions for a vertical cut, $\phi(x = a/2, y)$, $0 < y < a$.



[**2.**] Solve Laplace's equation in the same way as problem 1 except for the less symmetric region of Figure 2.





500 iterations
100 term analytic
φ(0.5 , y)
y
0
0.2
0.4
0.6
0.8
1
0
1
2
3
4

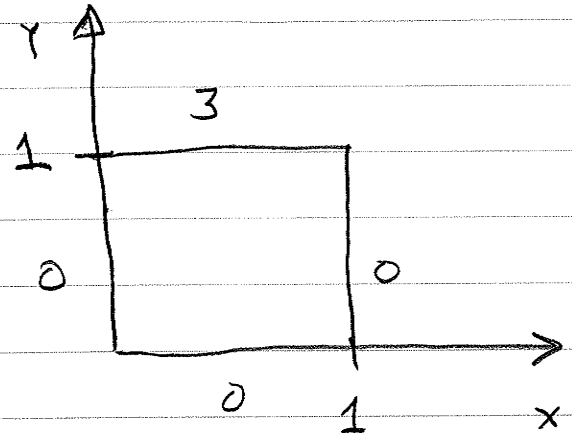
0.000	0.000000	0.000000
0.100	0.106220	0.105402
0.200	0.222586	0.221103
0.300	0.360037	0.358247
0.400	0.531028	0.529594
0.500	0.750000	0.750000
0.600	1.033178	1.036049
0.700	1.396787	1.403707
0.800	1.852191	1.862376
0.900	2.396460	2.405068
1.000	3.000000	3.009455

```
#include <stdio.h>
#include <math.h>

int main(void)

{

//  DECLARATIONS:

int i,j,it,Nit;
double phi[11][11],newphi[11][11],w;
double phianalytic[11],pi,coeff,sgn,logdiff;
FILE * fileout;

//  OPEN OUTPUT FILE:

fileout=fopen("laplace.dat","w");

w=1.0;
Nit=500;

//  ITERATIVE SOLUTION:

//      a.  INITIALIZE THE POTENTIAL

for (i=0;i<11;i++)
   {
for (j=0;j<11;j++)
   {
   phi[i][j]   =0.0;
   newphi[i][j]=0.0;
   }
   }

for (i=0;i<11;i++)
   {
   phi[i][10]=3.0;
   }

//      b.  ITERATION LOOP

for (it=1;it<Nit;it++)
   {

//      c.  CALCULATE NEW POTENTAIL

for (i=1;i<10;i++)
   {
for (j=1;j<10;j++)
   {
   newphi[i][j]=0.25*( phi[i-1][j]+phi[i+1][j]+phi[i][j-1]+phi[i][j+1] );
   }
   }

//      d.  RESET POTENTAIL TO NEW VALUE 0<w<1 IS MIXING PARAMETER

for (i=1;i<10;i++)
   {
for (j=1;j<10;j++)
```

```
 {
 phi[i][j]=(1.0-w)*phi[i][j]+w*newphi[i][j];
 }
 }


}

//     e.  WRITE POTENTIAL OUT

//  ANALYTIC SOLUTION:
//  GET FIRST 100 TERMS IN SUM
//  FANCY LOG STUFF TO AVOID OVERFLOW

pi=4.0*atan(1.0);
for (j=0; j<11; j=j+1)
{
phianalytic[j]=0.0;
sgn=1.0;
for (i=1; i<202; i=i+2)
{

coeff=12.0 / ( pi*i*sinh(pi*i) );
phianalytic[j]=phianalytic[j]+sgn*coeff*sinh(pi*i*j*0.1);
sgn=-sgn;

}
}

for (j=0; j<11; j=j+1)
{
fprintf(fileout,"%7.3lf %12.6lf %12.6lf \n",0.1*j,phi[5][j],phianalytic[j]);
}

fclose(fileout);

}
```

1.

HW #4 — Analytic soln



$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$$\phi = A(x)B(y)$$

$$A''B + AB'' = 0$$

$$\frac{A''}{A} = -k^2 = -\frac{B''}{B}$$

using also $\phi(x=0,y)=0$, $\phi(x,y=0)=0$

$$A(x) = \sin kx \qquad B(y) = \sinh ky$$

$$k = n\pi \quad \leftarrow \text{from } \phi(x=1,y)=0$$

$$\phi(x,y) = \sum_n \alpha_n \sin(n\pi x)\sinh(n\pi y)$$

$$3 = \phi(x,1) = \sum_n \alpha_n \sin(n\pi x)\sinh n\pi$$

$$3\int_0^1 \sin m\pi x\, dx = \alpha_m \sinh m\pi \left(\tfrac{1}{2}\right)$$

$$3\left(-\frac{\cos m\pi x}{m\pi}\right)\Bigg|_0^1 = \frac{3}{m\pi}\left[1-(-1)^m\right] = \begin{cases} 6/m\pi & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

2.

$$\phi(x,y) = \sum_{m=1,3,5} \frac{12}{m\pi \sinh(m\pi)} \sin(m\pi x)\sinh(m\pi y)$$

$$\phi(\tfrac{1}{2}, y) = \sum_{m=1,3,5} \frac{12}{m\pi \sinh(m\pi)} (-1)^{m+1} \sinh(m\pi y)$$

Check $y = 1$

$$\phi(\tfrac{1}{2}; 1) = \frac{12}{\pi}\left\{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots\right\}$$

↗

100 terms	0.78040	
200 "	0.78290	$\frac{\pi}{4} = .78540$
500 "	0.78440	
1000 "	0.78490	slowly converging !!

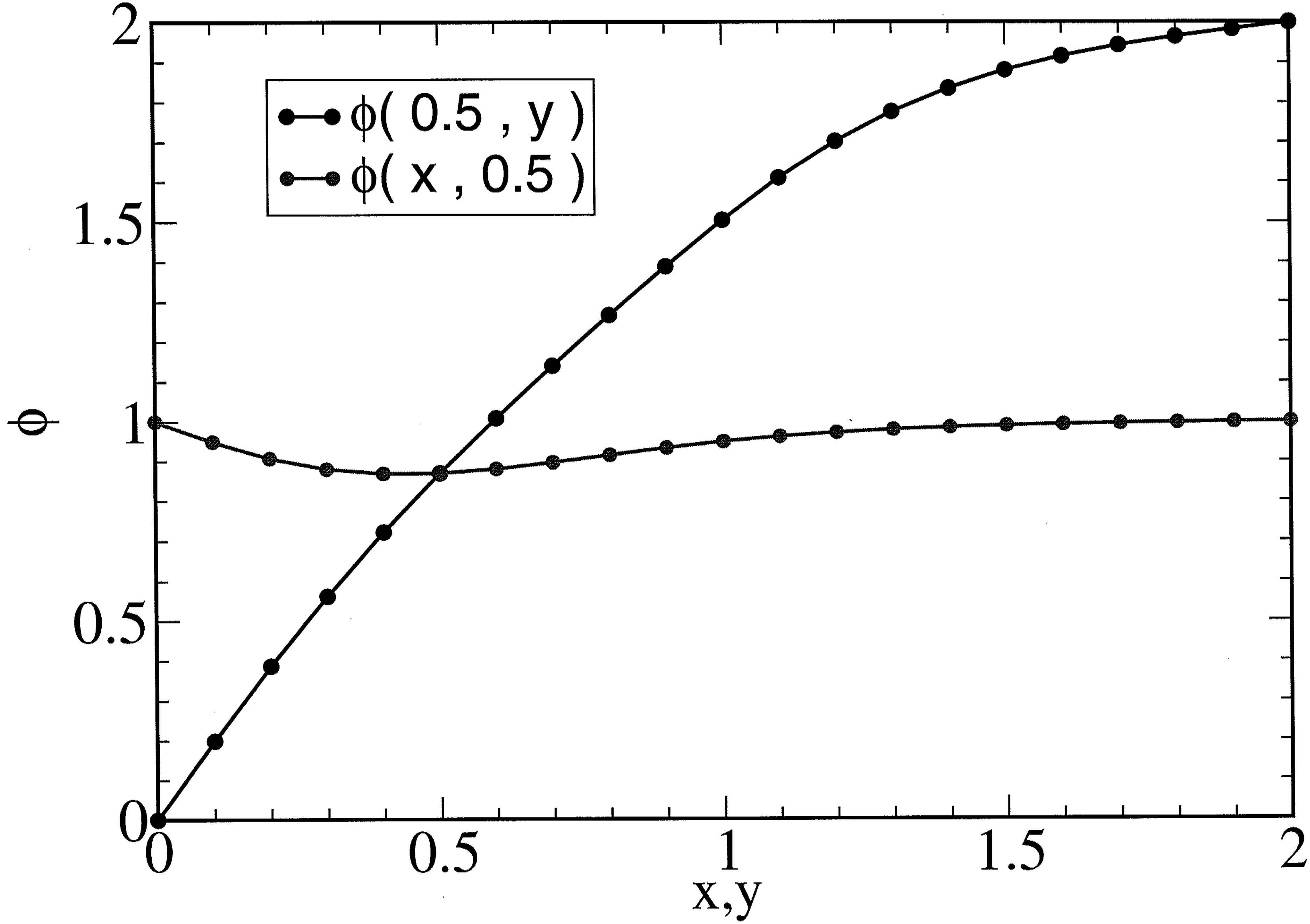
Controlling overflow:

$$\ln(\sinh x) = \ln\left(\frac{e^x - e^{-x}}{2}\right) = \ln e^x\left(\frac{1-e^{-2x}}{2}\right)$$

$$= x + \ln\left(\frac{1-e^{-2x}}{2}\right)$$

$$\frac{\sinh x'}{\sinh x} = \exp\left[\ln(\sinh x') - \ln(\sinh x)\right] = \exp\left[x' - \ln\left(\frac{1-e^{-2x'}}{2}\right) - x + \ln\left(\frac{1-e^{-2x}}{2}\right)\right]$$

$$= \exp\left[x' - x - \ln\left(\frac{1-e^{-2x'}}{1-e^{-2x}}\right)\right]$$


φ(0.5 , y)
φ(x , 0.5)
φ
x,y
0
0.5
1
1.5
2

vertical cut at x=0.5

0.000	0.000000
0.100	0.198433
0.200	0.387250
0.300	0.561793
0.400	0.721867
0.500	0.869646
0.600	1.008027
0.700	1.139654
0.800	1.266325
0.900	1.388459
1.000	1.504508
1.100	1.610640
1.200	1.701958
1.300	1.776492
1.400	1.835208
1.500	1.880596
1.600	1.915546
1.700	1.942774
1.800	1.964633
1.900	1.983125
2.000	2.000000

horizontal cut at y=0.5

0.000	1.000000
0.100	0.948471
0.200	0.906872
0.300	0.880108
0.400	0.868473
0.500	0.869646
0.600	0.880219
0.700	0.896532
0.800	0.915134
0.900	0.933185
1.000	0.948875
1.100	0.961608
1.200	0.971528
1.300	0.979083
1.400	0.984773
1.500	0.989051
1.600	0.992290
1.700	0.994789
1.800	0.996784
1.900	0.998467
2.000	1.000000

```
#include <stdio.h>
#include <math.h>

int main(void)

{

//  DECLARATIONS:

int i,j,it,Nit;
double phi[21][21],newphi[21][21],w;
FILE * fileout;

//  OPEN OUTPUT FILE:

fileout=fopen("laplace.dat","w");

w=1.0;
Nit=500;

//  ITERATIVE SOLUTION:

//      a.  INITIALIZE THE POTENTIAL

for  (i=0;i<21;i++)
   {
for  (j=0;j<21;j++)
   {
   phi[i][j]   =0.0;
   newphi[i][j]=0.0;
   }
   }

for  (j=0;j<21;j++)
   {
   phi[0][j]=1.0;
   }

for  (j=0;j<11;j++)
   {
   phi[20][j]=1.0;
   }

for  (j=10;j<21;j++)
   {
   phi[10][j]=3.0;
   }

for  (i=0;i<11;i++)
   {
   phi[i][20]=2.0;
   }

for  (i=10;i<21;i++)
   {
   phi[i][10]=2.0;
   }

//      b.  ITERATION LOOP
```

```
for (it=1;it<Nit;it++)
   {

//     c.  CALCULATE NEW POTENTAIL

for (i=1;i<10;i++)
   {
for (j=1;j<20;j++)
   {
   newphi[i][j]=0.25*( phi[i-1][j]+phi[i+1][j]+phi[i][j-1]+phi[i][j+1] );
   }
   }

for (i=10;i<20;i++)
   {
for (j=1;j<10;j++)
   {
   newphi[i][j]=0.25*( phi[i-1][j]+phi[i+1][j]+phi[i][j-1]+phi[i][j+1] );
   }
   }

//     d.  RESET POTENTAIL TO NEW VALUE 0<w<1 IS MIXING PARAMETER

for (i=1;i<10;i++)
   {
for (j=1;j<20;j++)
   {
   phi[i][j]=(1.0-w)*phi[i][j]+w*newphi[i][j];
   }
   }

for (i=10;i<20;i++)
   {
for (j=1;j<10;j++)
   phi[i][j]=(1.0-w)*phi[i][j]+w*newphi[i][j];
   {
   }
   }


 }

//     e.  WRITE POTENTIAL OUT

 for (j=0; j<21; j=j+1)
 {
 fprintf(fileout,"%7.3lf %12.6lf \n",0.1*j,phi[5][j]);
 }

 for (i=0; i<21; i=i+1)
 {
 fprintf(fileout,"%7.3lf %12.6lf \n",0.1*i,phi[i][5]);
 }


 fclose(fileout);

 }
```