Physics 204B, Winter 2011, Problem Set 1

[1.] In class we introduced the generating function of the Bessel polynomials,

\[ g(x, t) = \exp \left[ \frac{x}{2} (t - \frac{1}{t}) \right] = \sum_{n=-\infty}^{\infty} J_n(x) t^n. \]

By taking the derivative \( \partial / \partial t \) on both expressions for \( g \), derive the recurrence relation,

\[ J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x). \]

[2.] Show the Laplace transform of the zeroth order Bessel function,

\[ J_0(a) = \int_0^\infty e^{-ax} J_0(bx) \, dx = 1 / \sqrt{a^2 + b^2}. \]

Hint: One approach is to use the series expansion of \( J_0(x) \). (I used \( a \) for the transform variable to avoid confusion with the summation variable \( s \) used in class for the series index.)

[3.] Since the trigonometric functions form a complete set (the idea behind Fourier series) we should be able to expand the Bessel functions in terms of them. Show that this is indeed the case by proving

\[(a) \quad \cos x = J_0(x) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(x) \]

\[(b) \quad \sin x = 2 \sum_{n=1}^{\infty} (-1)^{n+1} J_{2n+1}(x) \]

Why would you expect the expansion of \( \cos(\sin) \) to involve only Bessel functions \( J_n(x) \) of even(odd) order \( n \)?

[4.] Consider a cylinder of radius \( a \) and height \( h \) with potential \( \psi = 0 \) on all surfaces except for the top end section, which has a given potential \( \psi(\rho, \phi) \). Write down an expression for the potential \( \psi(\rho, \phi, z) \) everywhere inside the cylinder in terms of an expansion in the appropriate functions. Include an equation for the expansion coefficients. Can you evaluate the coefficients when \( \psi(\rho, \phi) = \psi_0 \), a constant?

\[ g(x,t) = e^{\frac{x}{2}(t - \frac{x}{t^2})} = \sum J_n(x) t^n \]

\[ \frac{\partial}{\partial t} \frac{x}{2} \left(1 + \frac{1}{t^2}\right) e^{\frac{x}{2}(t - \frac{x}{t^2})} = \sum J_n(x) n t^{n-1} \]

\[ \frac{x}{2} \left(1 + \frac{1}{t^2}\right) \sum J_n(x) t^n = \sum J_n(x) n t^{n-1} \]

get every thing in terms of \( t^n \) by changing summation variable, eg on rhs let \( n \rightarrow n+1 \)

\[ \frac{x}{2} \left( J_n + J_{n+2} \right) = (n+1) J_{n+1} \]

If we now \( n+1 \rightarrow n \)

\[ \frac{x}{2} \left( J_{n-1} + J_{n+1} \right) = n J_n \]

So \( J_{n-1} + J_{n+1} = \frac{2n}{x} J_n \) \( \Box \)
Let's use the series approach

\[ J_n(x) = \sum_{s=0}^{\infty} \frac{1}{s! (s+n)!} (-1)^s \left( \frac{x}{2} \right)^{n+2s} \]

so

\[ J_0(x) = \sum_{s=0}^{\infty} \frac{1}{s!} \frac{1}{s!} \left( -1 \right)^s \left( \frac{x}{2} \right)^{2s} \]

\[ \therefore J_0(a) = \int_0^\infty e^{-ax} J_0(bx) \, dx \]

\[ = \int_0^\infty e^{-ax} \sum_{s=0}^{\infty} \frac{1}{s!} \frac{1}{s!} \left( -1 \right)^s \left( \frac{bx}{2} \right)^{2s} \, dx \]

Now

\[ \int_0^\infty e^{-ax} x^n \, dx = \frac{n!}{a^{n+1}} \]

\[ \therefore J_0(a) = \sum_{s=0}^{\infty} \frac{1}{s!} \frac{1}{s!} \left( -1 \right)^s \frac{b^{2s} (2s)!}{2^{2s} a^{2s+1}} \]

Working backwards now

\[ (a^2 + b^2)^{-1/2} = \left[ a^2 \left( 1 + b^2/a^2 \right) \right]^{-1/2} = \frac{1}{a} \left( 1 + \frac{b^2}{a^2} \right)^{-1/2} \]

\[ = \frac{1}{a} \left[ 1 - \frac{1}{2} \frac{b^2}{a^2} + \frac{1}{2} \frac{3}{2} \frac{1}{2} \left( \frac{b^2}{a^2} \right)^2 + \frac{1}{2} \frac{3}{2} \frac{5}{2} \frac{1}{2} \frac{3}{2} \left( \frac{b^2}{a^2} \right)^3 + \cdots \right] \]

\[ = \frac{1}{a} - \frac{b^2}{2a^3} + \frac{3}{8} \frac{b^4}{a^5} - \frac{15}{16} \frac{b^6}{a^7} + \cdots \]
Expanding $J_0(a) = \frac{1}{a} - \frac{2^2}{4^3} \frac{2!}{2^2} + \frac{1}{2!} \frac{\frac{1}{2} \frac{3}{2} \frac{4}{2}}{4^5} \frac{3!}{2^4} - \cdots$

\[
\frac{1}{a^4} \frac{1}{2^2} \frac{1 \cdot 2 \cdot 3 \cdot 4}{16} = \frac{3}{8}
\]

So things are working at low order.

To prove general order $s$ we need

\[
\frac{1}{s!} \frac{1}{s!} \frac{(2s)!}{2^{2s}} \quad \frac{? \quad \Rightarrow \quad \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right) \cdots \left(\frac{2s-1}{2}\right) \frac{1}{s!}}{\frac{(2s)!}{2^{2s}}}
\]

$s!$ terms cancel as do $2^s$

\[
\frac{1}{s!} \frac{(2s)!}{2^{2s}} \quad \frac{? \quad \Rightarrow \quad 1 \cdot 3 \cdot 5 \cdots (2s-1)}{\frac{(2s)!}{2^{2s}}}
\]

\[
(2s)! \quad \Rightarrow \quad 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots (2s-1)(2s)
\]

\[
\frac{s!}{2^s} \quad \Rightarrow \quad \frac{(1 \cdot 2 \cdot 3 \cdot 4 \cdots s)(2 \cdot 2 \cdots 2)}{(\frac{(2s)!}{2^{2s}})} \quad \checkmark
\]

all even terms go away!
Francesca suggests a straightforward approach:

Simply match power series expansion in $x$ on both sides,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

is well-known.

$$J_0(x) + 2 \sum_{m=1}^{\infty} \frac{(-1)^m J_{2m}(x)}{2m}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s}}{s! \cdot s!} + 2 \sum_{m=1}^{\infty} (-1)^m \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2m+2s}}{s! \cdot (2m+2s)!}$$

It is natural to define $n = m + s$ in second term so that

one gets a simple power $x^{2n}$ as in all other sums.

We eliminate the $s$ sum in favor of an "$n" sum,

For a given "$n" the variable "$m" runs from 1 to "$n"

(See figure). Thus second term * is

$$2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2m} \cdot (n-m)! \cdot (n+m)!}$$

$$S = n - m \quad s + 2m = n + m$$
Now all our power series have \((-1)^n x^{2n}\) (or \((-1)^{2n} x^{2n}\)) expansions and what we need to be true is that

\[
\frac{1}{(2n)!} = \frac{1}{2^{2n}} \frac{1}{n! \cdot n!} + 2 \sum_{m=1}^{n} \frac{1}{2^{2m}} \frac{1}{(n-m)! (n+m)!}
\]

The \(\cos x\) expansion, \(E_0(x)\) expansion, \(2 \sum E_{2n}(x)\) expansion after rearrangements in preceding page.

Multiplying through by \(2^{2n} (2n)!\) gives

\[
2^{2n} = \frac{(2n)!}{n! \cdot n!} + 2 \sum_{m=1}^{n} \frac{(2n)!}{(n-m)! (n+m)!}.
\]

If we define \(k = n-m\) so that \(m+n = 2n-k\)

\[
2^{2n} = \frac{(2n)!}{n! \cdot n!} + 2 \sum_{k=0}^{n-1} \frac{(2n)!}{k! (2n-k)!}.
\]

Consider row \(2n\) of Pascal's triangle. We know the elements sum to \(2^{2n}\) and the middle element is \((2n)!/n! n!\).

The sum of right is just the other elements summed up, so this again is "obvious". See next page.
\[
\begin{align*}
\text{row } 2n &= 6 & \text{ elements sum} \\
&= 2^n = 64 \\
\text{two identical sums} & \quad \sum_{k=0}^{n-1} \frac{(2n)!}{k!(2n-k)!}
\end{align*}
\]
A generating function proof is less algebra, but less obvious.

\[ e^{\lambda \left( t - \frac{1}{2} \right) t} = \sum_{n=-\infty}^{\infty} J_n(x) t^n \]

Let \( t = e^{i\theta} \)

\[ e^{i x \sin \theta} = \sum_{n=-\infty}^{\infty} J_n(x) e^{i\theta n} \]

\[ = J_0(x) + J_1(x) e^{i\theta} + J_{-1}(x) e^{-i\theta} \]

\[ + J_{-1}(x) e^{i\theta} + J_{-2}(x) e^{-2i\theta} + \ldots \]

Using \( J_n(x) = \frac{\sin(nx)}{\sin(x)} (-1)^n \)

\[ e^{i x \sin \theta} = J_0(x) + J_1(x) 2i \sin \theta + J_2(x) 2 \cos \theta + \ldots \]

\[ \cos(x \sin \theta) + i \sin(x \sin \theta) = J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x) \cos(n \theta) + 2i \sum_{n=0}^{\infty} J_{2n+1}(x) \sin(2n+1) \theta \]

If we choose \( \theta = \pi/2 \) and equate Real and Imaginary parts,

\[ \cos x = J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x) (-1)^n \]

\[ \sin x = 2 \sum_{n=0}^{\infty} J_{2n+1}(x) (-1)^n \]
In cylindrical coordinates

\[ \nabla^2 = \left[ \frac{1}{p} \frac{\partial}{\partial p} p \frac{\partial}{\partial p} + \frac{1}{p^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{z^2} \frac{\partial^2}{\partial z^2} \right] \]

We want to solve the Laplace eqn \( \nabla^2 V = 0 \)

Write \( V(p, \phi, z) = R(p) f(\phi) g(z) \) where

\[ \frac{1}{R} \frac{1}{p} \frac{\partial}{\partial p} p \frac{\partial R}{\partial p} + \frac{1}{p^2} \frac{1}{f} \frac{\partial^2 f}{\partial \phi^2} + \frac{1}{g} \frac{\partial^2 g}{\partial z^2} = 0 \]

Make the ansatz \( g = e^{\pm \alpha z} \) so \( \frac{1}{g} \frac{\partial^2 g}{\partial z^2} = \pm \alpha^2 \)

And also assume \( f(\phi) = e^{\pm i \mu \phi} \) so \( \frac{1}{f} \frac{\partial^2 f}{\partial \phi^2} = -\mu^2 \)

These will be justified by fact that they work! The remaining

eqn for \( R(p) \) is

\[ \frac{1}{R} \frac{\partial}{\partial p} p \frac{\partial R}{\partial p} + (\alpha^2 p^2 - \mu^2) R = 0 \]

whose soln is \( J_\mu(\alpha p) \),

So, we have success and a soln is

\[ J_\mu(\alpha p) e^{\pm i \mu \phi} e^{\pm \alpha z} \]

or \( \sinh \alpha z \frac{e^\alpha z}{\cosh \alpha z} \)
If we impose the condition $V(p, \phi, z=0) = 0$

The $z$ dependence must be $\sinh \alpha z$.

Furthermore, the requirement that $V(p = R, \phi, z) = 0$ means $J_m(\alpha R) = 0$ so $\alpha R$ must be a zero of $J_m$.

$$V(p, \phi, z) = \sum_{nm} a_{nm} J_m(\alpha_{nm} \phi) e^{i \alpha_{nm} z / R}$$

$\alpha_{nm}$ is $n$th root of $J_m$

The coefficients $a_{nm}$ are determined by the final set of boundary conditions on the top end of cylinder

$$\sum_{nm} a_{nm} J_m(\alpha_{nm} \frac{p}{R}) e^{i \alpha_{nm} z / R} = \Psi(p, \phi)$$

We can first use Fourier inversion to see that

$$\sum_{n} a_{nm} J_m(\alpha_{nm} \frac{p}{R}) \sinh \alpha_{nm} \frac{H}{R} = \int_{0}^{2\pi} \Psi(p, \phi) e^{-i \phi} \frac{df}{2\pi}$$
The final step is to use completeness/orthogonality of Bessel functions:

\[ \int_0^R J_m(\alpha_{nm} \frac{p}{R}) J_m(\alpha_{nm} \frac{p}{R}) p \, dp = \frac{R^2}{2 \pi} [J_{m+1}(\alpha_{nm})]^2 \]

\[ a_{nm} \sinh \alpha_{nm} \frac{H}{R} = \int_0^{2\pi} \frac{1}{2\pi} \int_0^R p \, dp \, \psi(p, \phi, z) e^{-i\alpha_{nm} \frac{p}{R}} J_m(\alpha_{nm} \frac{p}{R}) \frac{R^2}{2 \pi} [J_{m+1}(\alpha_{nm})]^2 \]

\[ a_{nm} = \frac{1}{2\pi} \frac{1}{\sinh \alpha_{nm} \frac{H}{R}} \int_0^{2\pi} \frac{1}{2\pi} \int_0^R p \, dp \, \psi(p, \phi) e^{i\alpha_{nm} \frac{p}{R}} J_m(\alpha_{nm} \frac{p}{R}) \]

\[ \psi(p, \phi, z) = \sum_{nm} a_{nm} J_m(\alpha_{nm} \frac{p}{R}) e^{i\phi} \sinh \alpha_{nm} \frac{z}{R} \]

The problem asks if we can get integrated above if \( \psi(p, \phi) = \Psi_0 \)

A constant \( \Psi_0 \) is clear that \( a_{nm} = 0 \) unless \( m = 0 \)

\[ a_{00} = \Psi_0 \frac{2}{R^2 [J_1(\alpha_{00})]^2} \frac{1}{\sinh \alpha_{00} \frac{H}{R}} \int_0^R p \, dp \, J_0(\alpha_{00} \frac{p}{R}) \]

Can that last integral be done?
We desire to evaluate \( \int_0^R g y p J_0 (\alpha_n x \frac{R}{\mu}) \) 

In Gradshetyn-Ryzhik we find 6.561.5

\[
\int_0^1 x^{u+1} J_v (a x) \, dx = \frac{1}{a} J_{v+1} (a) \quad \forall \nu \text{ with } Re > -1
\]

Setting \( u = 0 \)

\[
\int_0^1 x J_0 (a x) \, dx = \frac{1}{a} J_1 (a)
\]

Change variables in our integral \( p = R x \) so that

\[
\Rightarrow \int_0^1 R^2 x \, dx \, J_0 (\alpha_{n_0} x) = R^2 \frac{1}{\alpha_{n_0}} J_1 (\alpha_{n_0})
\]

So for the special case of \( \Psi = \Psi_0 \) constant on the top

\[
a_{nm} = \delta_{m,0} \Psi_0 \frac{2}{R^2} \frac{1}{(J_0 (\alpha_{n_0}))^2} \frac{1}{\sinh \frac{\alpha_{n_0} R}{\mu}} R^2 \frac{1}{\alpha_{n_0}} J_1 (\alpha_{n_0})
\]

Only \( m = 0 \) simplifies to

\[
a_{nm} = \delta_{m,0} 2 \Psi_0 \frac{1}{J_1 (\alpha_{n_0}) \sinh \frac{\alpha_{n_0} R}{\mu} \alpha_{n_0}}
\]