Physics 204B, Winter 2011, Problem Set 1

[1.] In class we introduced the generating function of the Bessel polynomials,

$$g(x,t) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(x) t^n \ .$$

By taking the derivative $\partial/\partial t$ on both expressions for g, derive the recurrence relation,

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \ .$$

[2.] Show the Laplace transform of the zeroth order Bessel function,

$$\mathcal{J}_0(a) = \int_0^\infty e^{-xa} J_0(bx)\, dx = 1 / \sqrt{a^2+b^2} \ .$$

Hint: One approach is to use the series expansion of $J_0(x)$. (I used a for the transform variable to avoid confusion with the summation variable s used in class for the series index.)

[3.] Since the trigonometric functions form a complete set (the idea behind Fourier series) we should be able to expand the Bessel functions in terms of them. Show that this is indeed this the case by proving

$$(a) \qquad \cos x = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$$

$$(b) \qquad \sin x = 2\sum_{n=1}^{\infty}(-1)^{n+1} J_{2n+1}(x)$$

Why would you expect the expansion of cos(sin) to involve only Bessel functions $J_n(x)$ of even(odd) order n?

[4.] Consider a cylinder of radius a and height h with potential $\psi = 0$ on all surfaces except for the top end section, which has a given potential $\psi(\rho,\phi)$. Write down an expression for the potential $\psi(\rho,\phi,z)$ everywhere inside the cylinder in terms of an expansion in the appropriate functions. Include an equation for the expansion coefficients. Can you evaluate the coefficients when $\psi(\rho,\phi) = \psi_0$, a constant?

[5.] Stone and Goldbart Problem 8-13.

$\boxed{1-1}$

$$g(x\,t) = e^{x/2\,(t - 1/t)} = \sum J_n(x)\, t^n$$

$\partial/\partial t$ $$\frac{x}{2}\left(1 + 1/t^2\right) e^{x/2\,(t-1/t)} = \sum J_n(x)\, n\, t^{n-1}$$

$$\frac{x}{2}\left(1 + 1/t^2\right) \sum J_n(x) t^n = \sum J_n(x)\, n\, t^{n-1}$$

get every thing in terms of t^n by changing summation variable, eg on rhs let $n \to n+1$

$$\frac{x}{2}\left(J_n + J_{n+2}\right) = (n+1)\, J_{n+1}$$

If we now $n+1 \to n$

$$\frac{x}{2}\left(J_{n-1} + J_{n+1}\right) = n\, J_n$$

so $$J_{n-1} + J_{n+1} = \frac{2n}{x} J_n$$ ▨

$\boxed{1-2}$ Let's use the series approach

$$J_n(x) = \sum_s \frac{1}{s!(s+n)!}(-1)^s\left(\frac{x}{2}\right)^{n+2s}$$

so $$J_0(x) = \sum_s \frac{1}{s!}\frac{1}{s!}(-1)^s\left(\frac{x}{2}\right)^{2s}$$

$$\therefore \quad \tilde{J}_0(a) = \int_0^\infty e^{-xa} J_0(bx)\,dx$$

$$= \int_0^\infty e^{-xa}\sum_s \frac{1}{s!}\frac{1}{s!}(-1)^s\left(\frac{bx}{2}\right)^{2s} dx$$

Now $$\int_0^\infty e^{-ax}x^n\,dx = n!/a^{n+1}$$

$$\tilde{J}_0(a) = \sum_s \frac{1}{s!}\frac{1}{s!}(-1)^s \frac{b^{2s}}{2^{2s}}\frac{(2s)!}{a^{2s+1}}$$

Working backwards now

$$(a^2+b^2)^{-1/2} = \left[a^2\left(1+b^2/a^2\right)\right]^{-1/2} = \frac{1}{a}\left(1+\frac{b^2}{a^2}\right)^{-1/2}$$

$$= \frac{1}{a}\left[1 - \frac{1}{2}\frac{b^2}{a^2} + \frac{1}{2}\frac{3}{2}\frac{1}{2}\left(\frac{b^2}{a^2}\right)^2 - \frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{1}{2\cdot3}\left(\frac{b^2}{a^2}\right)^3 \ldots\right]$$

$$= \frac{1}{a} - \frac{b^2}{2a^3} + \frac{3}{8}\frac{b^4}{a^5} - \frac{15}{16}\frac{b^6}{a^7} + \ldots$$

1-2 cont'd -1

$$\text{Expanding } \tilde{J}_0(a) = \frac{1}{a} - \frac{b^2}{a^3}\,\overset{1/2}{\frac{2!}{2^2}} + \underbrace{\frac{1}{2!}\frac{1}{2!}\frac{b^4}{a^5}\frac{4!}{2^4}}_{\frac{1}{2}\frac{1}{2}\frac{1\cdot2\cdot3\cdot4}{16} = \frac{3}{8}} - \ldots$$

So things are working at low order

To prove general order s we need

$$\frac{1}{s!}\frac{1}{s!}\frac{(2s)!}{2^{2s}} \stackrel{?}{=} \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\ldots\left(\frac{2s-1}{2}\right)\frac{1}{s!}$$

s! terms cancel as do 2^s

$$\frac{1}{s!}\frac{(2s)!}{2^s} \stackrel{?}{=} 1\cdot3\cdot5\cdots(2s-1)$$

$$\Downarrow$$

$$\begin{array}{ll} (2s)! \Rightarrow & 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdots(2s-1)(2s) \\ \hline s!\,2^s \Rightarrow & (1\cdot2\cdot3\cdot4\cdots s)(2\cdot2\cdot2\cdots2) \end{array} \quad \checkmark\checkmark$$

all even terms go away!

$\boxed{1-3}$ Francesca suggests a straightforward approach:

Simply match power series expansion in x on both sides

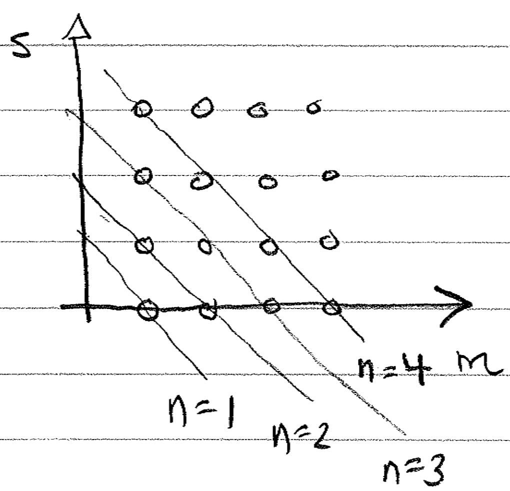
$$\cos x = \sum_{0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{is well-known}$$

$$J_0(x) + 2\sum_{m=1}^{\infty} (-1)^m J_{2m}(x)$$

$$= \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{2s} \frac{1}{s!\,s!} + 2 \underbrace{\sum_{m=1}^{\infty} (-1)^m \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{2m+2s} \frac{1}{s!\,(2m+s)!}}_{*}$$

It is natural to define $n = m + s$ in second term so that

one gets a simple power x^{2n} as in all other sums



We eliminate the "s" sum in favor of an "n" sum.
For a given "n" the variable "m" runs from 1 to "n" (See figure). Thus second term $*$ is

$$2\sum_{n=1}^{\infty} \sum_{m=1}^{n} (-1)^n \frac{x^{2n}}{2^{2n}} \frac{1}{\underbrace{(n-m)!}_{s=n-m}\,\underbrace{(n+m)!}_{s+2m=n+m}}$$

1-3 cont'd -1 Now all our power series have $(-1)^n x^{2n}$ (or $(-1)^s x^{2s}$) expansions and what we need to be true is that

$$\frac{1}{(2n)!} \stackrel{?}{=} \frac{1}{2^{2n}}\frac{1}{n!\,n!} + 2\sum_{m=1}^{n} \frac{1}{2^{2n}} \frac{1}{(n-m)!\,(n+m)!}$$

$\cos x$ expansion $J_0(x)$ expansion $2\sum J_{2n}(x)$ expansion after rearrangements on preceding page

Multiplying through by $2^{2n}(2n)!$ gives

$$2^{2n} \stackrel{?}{=} \frac{(2n)!}{n!\,n!} + 2\sum_{m=1}^{n} \frac{(2n)!}{(n-m)!\,(n+m)!}$$

If we define $k = n-m$ so that $n+m = 2n-k$

$$2^{2n} \stackrel{?}{=} \frac{(2n)!}{n!\,n!} + 2\sum_{k=0}^{n-1} \frac{(2n)!}{k!\,(2n-k)!}$$

Consider row $2n$ of Pascal's triangle. We know the elements sum to 2^{2n} and the middle element is $(2n)!/n!\,n!$. The sum at right is just the other elements summed up, so this eqn is "obvious". See next page.

1-3 cont'd - 2

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1 ← row $2n=6$ elements sum to $2^6=64$

$\frac{6!}{3!\,3!}$

two identical sums $\sum_{k=0}^{n-1} (2n)!/k!(2n-k)!$

cont'd

1-3-3 A generating function proof is less algebra, but less obvious

$$e^{x/2(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

Let $t = e^{i\theta}$

$$e^{x/2\, i x \sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x) e^{i\theta n}$$

$$= J_0(x) + J_1(x) e^{i\theta} + J_{-1}(x) e^{-i\theta}$$

$$+ J_2(x) e^{2i\theta} + J_{-2}(x) e^{-2i\theta} + \dots$$

using $J_{-n}(x) = J_n(x)(-1)^n$

$$e^{ix\sin\theta} = J_0(x) + J_1(x)\, 2i\sin\theta + J_2(x)\, 2\cos 2\theta + \dots$$

$$\cos(x\sin\theta) + i\sin(x\sin\theta) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos 2n\theta + 2i\sum_{n=0}^{\infty} J_{2n+1}(x)\sin(2n+1)\theta$$

If we choose $\theta = \pi/2$ and equate Real and Imag parts...

$$\cos x = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)(-1)^n$$

$$\sin x = 2\sum_{n=0}^{\infty} J_{2n+1}(x)(-1)^n$$

$\boxed{1-4}$ In cylindrical coordinates

$$\nabla^2 = \left[\frac{1}{\rho} \partial/\partial\rho \, \rho \partial/\partial\rho + \frac{1}{\rho^2} \partial^2/\partial\phi^2 + \partial^2/\partial z^2 \right]$$

We want to solve the Laplace eqn $\nabla^2 V = 0$

write $V(\rho, \phi, z) \equiv R(\rho) f(\phi) G(z)$ whence

$$\frac{1}{R} \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial R}{\partial \rho} + \frac{1}{\rho^2} \frac{1}{f} \frac{\partial^2 f}{\partial \phi^2} + \frac{1}{G} \frac{\partial^2 G}{\partial z^2} = 0$$

Make the ansatz $G = e^{\pm \alpha z}$ so $\frac{1}{G} \partial^2 G/\partial z^2 = +\alpha^2$

and also assume $f(\phi) = e^{\pm i m \phi}$ so $\frac{1}{f} \frac{\partial^2 f}{\partial \phi^2} = -m^2$

These will be justified by fact that they work! The remaining

eqn for $R(\rho)$ is

$$\rho \frac{1}{R} \frac{\partial}{\partial \rho} \rho R + (\alpha^2 \rho^2 - m^2) R = 0$$

whose sol'n is $J_m(\alpha \rho)$,

So, we have success and a soln is

$$J_m(\alpha \rho) e^{\pm i m \phi} e^{\pm \alpha z}$$

or $\sinh \alpha z$
$\cosh \alpha z$

If we impose the condition $V(\rho, \phi, z=0) = 0$

The z dependence must be $\sinh \alpha z$.

Furthermore, the requirement that $V(\rho = R, \phi, z) = 0$

means $J_m(\alpha R) = 0$ so αR must be a zero of J_m.

$$V(\rho, \phi, z) = \sum_{nm} a_{nm} J_m(\alpha_{nm} \frac{\rho}{R}) e^{im\phi} \sinh \alpha_{nm} z/R$$

α_{nm} is nth root of J_m

The coefficients a_{nm} are determined by the final

set of boundary conditions on the top end of cylinder

$$\sum_{nm} a_{nm} J_m(\alpha_{nm} \frac{\rho}{R}) e^{im\phi} \sinh \alpha_{nm} \frac{H}{R} = \psi(\rho, \phi)$$

We can first use Fourier inversion to see that

$$\sum_{n} a_{nm} J_m(\alpha_{nm} \frac{\rho}{R}) \sinh \alpha_{nm} \frac{H}{R} = \int_0^{2\pi} \psi(\rho, \phi) e^{-im\phi} \frac{d\phi}{2\pi}$$

The final step is to use completeness/orthogonality of Bessel functions:

$$\int_0^R J_m\left(\alpha_{nm}\frac{\rho}{R}\right) J_m\left(\alpha_{n'm}\,\rho/R\right)\rho\,d\rho = \delta_{nm}\frac{R^2}{2}\left[J_{m+1}(\alpha_{nm})\right]^2$$

$$a_{nm}\sinh\alpha_{nm}\frac{H}{R} = \int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^R \frac{\rho\,d\rho}{\frac{R^2}{2}\left[J_{m+1}(\alpha_{nm})\right]^2}\,\psi(\rho,\phi)\,e^{-im\phi}\,J_m\left(\alpha_{nm}\frac{\rho}{R}\right)$$

$$\boxed{\begin{aligned} a_{nm} &= \frac{1}{2\pi}\,\frac{1}{\frac{R^2}{2}\left[J_{m+1}(\alpha_{nm})\right]^2}\,\frac{1}{\sinh\alpha_{nm}\frac{H}{R}}\int_0^{2\pi}d\phi\int_0^R\rho\,d\rho\,\psi(\rho,\phi)\,e^{-im\phi}\,J_m\left(\alpha_{nm}\frac{\rho}{R}\right) \\ \psi(\rho,\phi,z) &= \sum_{nm} a_{nm}\,J_m\left(\alpha_{nm}\,\rho/R\right)e^{im\phi}\sinh\alpha_{nm}\frac{z}{R}\end{aligned}}$$

The problem asks if we can get integrals done if $\psi(\rho,\phi) = \psi_0$ a constant. It is clear that $a_{nm} = 0$ unless $m = 0$

$$a_{n0} = \psi_0\,\frac{2}{R^2\left[J_1(\alpha_{n0})\right]^2}\,\frac{1}{\sinh\alpha_{n0}\frac{H}{R}}\int_0^R\rho\,d\rho\,J_0\left(\alpha_{n0}\frac{\rho}{R}\right)$$

Can that last integral be done?

1-4 cont'd -4

We desire to evaluate $\int_0^R \rho\, d\rho\, J_0(\alpha_{n0}\, \rho/R)$

In Gradshteyn-Ryzhik we find 6.561.5

$$\int_0^1 x^{\nu+1} J_\nu(ax)\, dx = \frac{1}{a} J_{\nu+1}(a) \qquad \forall \nu \text{ with } \mathrm{Re}\,\nu > -1$$

setting $\nu = 0$

$$\int_0^1 x\, J_0(ax)\, dx = \frac{1}{a} J_1(a)$$

Change variables in our integral $\rho = Rx$ so that

$$\rightarrow \int_0^1 R^2\, x\, dx\, J_0(\alpha_{n0}\, x) = R^2 \frac{1}{\alpha_{n0}} J_1(\alpha_{n0})$$

So for the special case of $\psi = \psi_0$ constant on the top

$$a_{nm} = \delta_{m,0}\, \psi_0\, \frac{2}{R^2} \frac{1}{(J_1(\alpha_{n0}))^2} \frac{1}{\sinh \alpha_{n0} \frac{H}{R}}\, R^2 \frac{1}{\alpha_{n0}} J_1(\alpha_{n0})$$

only m=0 simplifies to

$$a_{nm} = \delta_{m,0}\, 2\psi_0 \frac{1}{J_1(\alpha_{n0}) \sinh \alpha_{n0} \frac{H}{R}\, \alpha_{n0}}$$