Let's now set $k=0$ so we are considering the Laplace eqn instead of the more general Helmholtz eqn.

We will come back to Helmholtz where we will see radial eqn leads to Bessel functions. But $k^2=0$ is much simpler so focus on it for now.

Even so, there is a subtle point. This is the fact that Q cannot be arbitrary but must equal $l(l+1)$ where l is an integer!

Sort of analogous to restriction that $m=$ integer so that $P(\phi+2\pi)=P(\phi)$

Let $x=\cos\theta$ $T(\theta)\to T(x)$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dT}{d\theta}+\left(Q-\frac{m^2}{\sin^2\theta}\right)T=0$$

$$\frac{d}{d\theta}=\frac{dx}{d\theta}\frac{d}{dx}=-\sin\theta\frac{d}{dx}= \qquad \sin\theta=\sqrt{1-x^2}$$

$$-\frac{1}{\sqrt{1-x^2}}\sqrt{1-x^2}\frac{d}{dx}\sqrt{1-x^2}\frac{dT}{dx}+\left(Q-\frac{m^2}{1-x^2}\right)T=0$$

$$-(1-x^2)\left[\frac{d^2T}{dx^2}+\frac{2x}{\sqrt{1-x^2}}\frac{dT}{dx}\right]+\left(Q-\frac{m^2}{1-x^2}\right)T=0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) + \left(Q - \frac{m^2}{\sin^2\theta}\right)T = 0$$

$$x = \cos\theta \qquad \sin\theta = \sqrt{1-x^2}$$

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx}$$

$$+\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}T\right] + \left(Q - \frac{m^2}{1-x^2}\right)T = 0$$

$$(1-x^2)\frac{d^2T}{dx^2} - 2x\frac{dT}{dx} + \left(Q - \frac{m^2}{1-x^2}\right)T = 0$$

Let's consider in detail the case when $Q = \ell(\ell+1)$.

Defining $P(x) = (1-x^2)^{-m/2}\,T(x)$ yields

$$* \quad (1-x^2)\frac{d^2P}{dx^2} - 2(m+1)x\frac{dP}{dx} + (\ell-m)(\ell+m+1)P = 0$$

For $m=0$

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \ell(\ell+1)P = 0$$

$P = P_\ell =$ Legendre polynomials

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$$T(x) = (1-x^2)^{m/2}\, p(x)$$

$$T'(x) = (1-x^2)^{m/2-1}\,\frac{m}{2}(-2x)\, p(x) + (1-x^2)^{m/2}\, p'(x)$$

$$T''(x) = (1-x^2)^{m/2-2}\left(\frac{m}{2}-1\right)(-mx)\, p(x)$$

$$+ (1-x^2)^{m/2-1}(-m)\, p(x)$$

$$+ (1-x^2)^{m/2-1}(-mx)\, p'(x)$$

$$+ (1-x^2)^{m/2-1}\,\frac{m}{2}(-2x)\, p'(x)$$

$$+ (1-x^2)^{m/2}\, p''(x)$$

$$(1-x^2)T'' - 2xT' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]T = 0$$

$$(1-x^2)^2 T'' - 2x(1-x^2)T' + \left(\ell(\ell+1)(1-x^2) - m^2\right)T = 0$$

$$(1-x^2)^{m/2}\left(\frac{m}{2}-1\right)(-mx)\, p(x) + (1-x^2)^{m/2}(-2x)(-mx)\, p(x)$$

$$+ (1-x^2)^{m/2}(1-x^2)(-m)\, p(x) + (1-x^2)^{m/2}(-2x)\, p'(x)$$

$$+ (1-x^2)^{m/2}(1-x^2)(-mx)\, p'(x) + \left[\ell(\ell+1)(1-x^2) - m^2\right](1-x^2)^{m/2}\, p(x)$$

$$+ (1-x^2)^{m/2}(1-x^2)(-mx)\, p'(x) \qquad = 0$$

$$+ (1-x^2)^{m/2}(1-x^2)\, p''(x)$$

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cancelling $(1-x^2)^{m/2}$

$\cancel{(\frac{m}{2}-1)(-mx)}$

$$p(x)\left[\left(\frac{m}{2}-1\right)(-mx)+(1-x^2)(-m)-2x(-mx)+l(l+1)(1-x^2)-m^2\right]$$

$$p'(x)\left[2(1-x^2)(-mx)-2x\right]$$

$$p''(x)\left[(1-x^2)\right]$$

$$\ell^2 + \ell - 2 = (\ell-1)(\ell+2)$$

$$\frac{d}{dx}: \quad (1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \ell(\ell+1)P = 0$$

$$-2x\,P'' + (1-x^2)\,P''' - 2P' - 2xP'' + \ell(\ell+1)P' = 0$$

$$(1-x^2)\frac{d^2}{dx^2}(P') - 2(1+1)x\frac{d}{dx}P' + (\ell-1)(\ell+1+1)P' = 0$$

$$(1-x^2)\frac{d^2}{dx^2}(P') - 2(m+1)x\frac{d}{dx}P' + (\ell-m)(\ell+m+1)P' = 0$$

So our sol'ns for $m \neq 0$ are obtained from the $m=0$ solns by differentiation!

$$T(x) = P_\ell^m(x) = \cancel{(-1)^m}(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x)$$ Associated Legendre Functions

are solns to the θ part of the Laplace eqn

$$S(\phi) = e^{\pm im\phi}$$

Finally, radial part

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{dR}{dr} + \cancelto{0}{k^2R} - \frac{\ell(\ell+1)R}{r^2} = 0$$

$$\frac{d}{dr}r^2\frac{dR}{dr} - \ell(\ell+1)R = 0$$

$$R = r^\ell \qquad \frac{dR}{dr} = \ell r^{\ell-1} \qquad r^2\frac{dR}{dr} = \ell r^{\ell+1}$$

$$R = r^{-(\ell+1)} \qquad \frac{dR}{dr} = -(\ell+1)r^{-(\ell+2)} \qquad r^2\frac{dR}{dr} = -(\ell+1)r^{-\ell}$$

Soln to Laplace Eqn in Spherical coordinates

$R(r)\,T(\cos\theta)\,S(\phi)$

$R_\ell(r) = r^\ell$ or $r^{-(\ell+1)}$

$S_m(\phi) = e^{im\phi}$

$x=\cos\theta \qquad T(x) = P_\ell^m(x) = \cancel{(-1)}^m (1-x^2)^{m/2} \dfrac{d^m}{dx^m} P_\ell(x)$ $\quad (1-x^2)^{m/2}$

$P_\ell^0(x) = P_\ell(x)$

~~We never said why we forced $\alpha = \ell(\ell+1)$~~ !

$P_1^0(x) = P_1(x) = x = \cos\theta$

$P_1^1(x) = (1-x^2)^{1/2} \dfrac{d}{dx} \underbrace{P_1(x)}_{x} = (1-x^2)^{1/2} = \sin\theta$

$P_2^0(x) = P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

$P_2^1(x) = \cancel{(-1)}(1-x^2)^{1/2} \dfrac{d}{dx}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$

$= \cancel{(-1)}(1-x^2)^{1/2}(3x)$

$= 3\cos\theta\sin\theta$

$P_2^2(x) = (1-x^2)^1 \dfrac{d^2}{dx^2}\left(3/2\,x^2 - 1/2\right) = (1-x^2)^1\,3$

$= 3\sin^2\theta$

One often combines θ, ϕ dependence into spherical Harmonics

$$Y_\ell^m(\theta,\phi) = (-1)^m \left[\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\cos\theta)\, e^{im\phi}$$

normalization factor

$$Y_0^0 = \frac{1}{2\sqrt{\pi}}$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \quad Y_\ell^m \, Y_{\ell'}^{m'} = \delta_{\ell\ell'}\delta_{mm'}$$

$$Y_1^0 = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$$

$$Y_1^{\pm 1} = \mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{\pm i\phi}$$

$$Y_2^0 = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(2\cos^2\theta - \sin^2\theta\right)$$

$$Y_2^{\pm 1} = \mp\frac{1}{2}\sqrt{\frac{15}{2\pi}}\cos\theta\sin\theta\, e^{\pm i\phi}$$

$$Y_2^{\pm 2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{\pm 2i\phi}$$

$$r^\ell \qquad r^{(-(\ell+1))}$$

Summary

To solve Laplace's Eqn, take linear combinations of

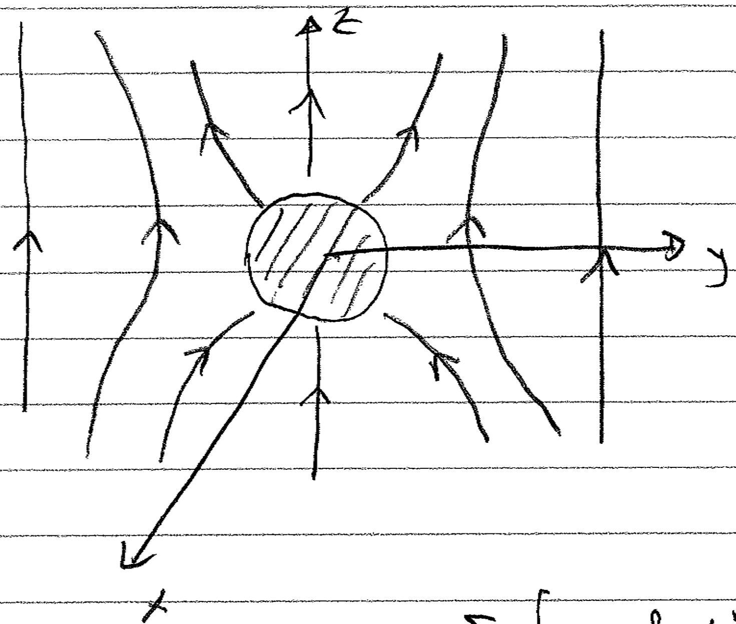
$$V(r,\theta,\phi) = \sum \quad r^{\ell} Y_{\ell}^{m}(\theta,\phi) \qquad r^{-(\ell+1)} Y_{\ell}^{m}(\theta,\phi)$$

coefficients will be chosen to fit boundary conditions

To solve Helmholtz Eqn: r^{ℓ} $r^{-(\ell+1)}$

get more complex (Bessel Eqns)

conducting sphere in a uniform E field



No ϕ dependence by symmetry

$$\nabla^2 V = 0$$

$$V = \sum_{\ell,m} \left[a_{\ell m} r^\ell Y_\ell^m(\theta,\phi) + b_{\ell m} r^{-(\ell+1)} Y_\ell^m(\theta,\phi) \right]$$

No ϕ dependence : $m=0$ use only P_ℓ the Legendre funct

$$= \sum_\ell a_\ell r^\ell P_\ell(\cos\theta) + b_\ell r^{-(\ell+1)} P_\ell(\cos\theta)$$

$r \to \infty$ $\quad \vec{E} = E_0 \hat{z} = -\vec{\nabla} V$

$$V = -E_0 z = -E_0 r\cos\theta = -E_0 r P_1(\cos\theta)$$

$$a_\ell = 0 \quad \ell > 1 \qquad a_1 = -E_0$$

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$r = r_0$

∴ sphere is an equipotential, choose it to be $V = 0$

$$-E_0 r_0 P_1(\cos\theta) + \sum_\ell b_\ell r_0^{-(\ell+1)} P_\ell(\cos\theta) = 0$$

But the P_ℓ are linearly independent so $b_\ell = 0$ except for b_1 and, in fact

$$-E_0 r_0 + b_1 r_0^{-2} = 0$$

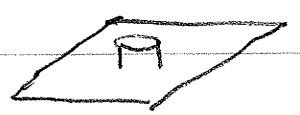
$$b_1 = E_0 r_0^3$$

$$V = \left[-E_0 r + \frac{E_0 r_0^3}{r^2}\right] P_1(\cos\theta)$$

$\boxed{\text{Compute charge distribution on sphere}}$

Gauss' law $\Phi = q/\epsilon_0$

$$E_{normal} \cancel{A} = \sigma \cancel{A}/\epsilon_0$$

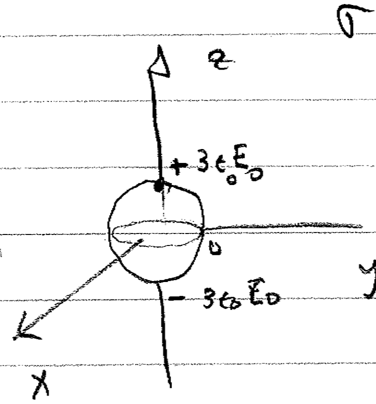


$$-\frac{\partial V}{\partial r} = \sigma/\epsilon_0$$

$P_1(\cos\theta)$

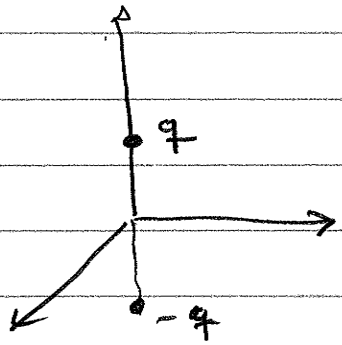
$$\sigma = -\epsilon_0 \left.\partial V/\partial r\right|_{r=r_0} = -\epsilon_0(-E_0 - 2E_0)\cos\theta$$

$$\sigma = 3\epsilon_0 E_0 \cos\theta$$



The expression $\frac{E_0 r_0^3}{r^2} \cos\theta$

is precisely the potential due to an electric dipole



$$V_+(r) = \frac{q}{4\pi\epsilon_0} \frac{1}{(r^2+a^2-2ar\cos\theta)^{1/2}}$$

generating function

$$= \frac{q}{4\pi\epsilon_0 r} \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n$$

$$V_-(r) = \frac{-q}{4\pi\epsilon_0 r} \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{-a}{r}\right)^n$$

$n=0$ cancels

$$n=1 \quad V(r) = \frac{+q}{4\pi\epsilon_0 r} P_1(\cos\theta)\left(\frac{a}{r}\right) 2$$

$$V(r) = \frac{2q}{4\pi\epsilon_0} \frac{a}{r^2} \cos\theta = \frac{p}{4\pi\epsilon_0 r^2} \cos\theta$$

we have $E_0 r_0^3 \frac{\cos\theta}{r^2}$

leading us to identify $\frac{p}{4\pi\epsilon_0} = E_0 r_0^3$

$$\boxed{p = 4\pi r_0^3 \epsilon_0 E_0}$$

completeness of spherical harmonics

$$f(\theta,\phi) = \sum_{mn} a_{mn} Y_\ell^m(\theta,\phi)$$

$$a_{\ell m} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \; Y_\ell^{m*}(\theta,\phi) f(\theta,\phi)$$

because $\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \; Y_\ell^{m*}(\theta,\phi) Y_{\ell'}^{m'}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}$

If there is no ϕ dependence

$$f(\theta) = \sum_n a_\ell Y_\ell^0(\theta,\phi)$$

$$Y_\ell^0(\theta,\phi) = P_\ell^0(\cos\theta)\cancel{e^{i0\phi}}$$

why is $Q = \ell(\ell+1)$

$$\left[(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + Q - \frac{m^2}{1-x^2} \right] T(x) = 0$$

$$* \begin{cases} \mathcal{L}\, T(x) = Q\, T(x) \\ \text{with} \quad \mathcal{L} = (1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} - \frac{m^2}{1-x^2} \quad \leftarrow \text{Hermitial} \end{cases}$$

In general we have T_{QM} $\perp$ & complete

$$** \qquad \int_{-1}^{1} dx\, T_{QM}(x)\, T_{Q'M}(x) = \delta_{Q,Q'}$$

Consider $Q \neq \ell(\ell+1)$

$$T_{QM}(\cos\theta)\, e^{im'\phi} = \sum_{\ell m} a_{\ell m}\, Y_\ell^m(\theta,\phi)$$

$$a_{\ell m} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, \underbrace{Y_\ell^{m*}(\theta,\phi)}_{\sim e^{-im'\phi}}\, T_{QM}(\cos\theta)\, e^{im'\phi}$$

so $m = m'$

$Y_\ell^m(\theta,\phi)$ is soln of $*$

so by $**$ integral gives zero