-1

HW 12.1.1, 12.1.2, 12.1.6
12.2.1, 12.2.2, 12.2.10

Legendre Functions

Generating function" $\to$ $g(t,x) \equiv (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$

Two approaches to special functions in physics
① PDE separate variables etc find in solving get these special function
② "generating function"

Since rhs is expansion in powers of t do so also for lhs

$$(1+u)^n = 1 + nu + \tfrac{1}{2}n(n-1)u^2 + \tfrac{1}{6}n(n-1)(n-2)u^3 + \dots$$

is binomial theorem

$$(1-2xt+t^2)^{-1/2} = 1 - \tfrac{1}{2}(-2xt+t^2)$$
$$+ \tfrac{1}{2}(-\tfrac{1}{2})(-\tfrac{3}{2})(-2xt+t^2)^2$$
$$+ \tfrac{1}{6}(-\tfrac{1}{2})(-\tfrac{3}{2})(-\tfrac{5}{2})(-2xt+t^2)^3$$

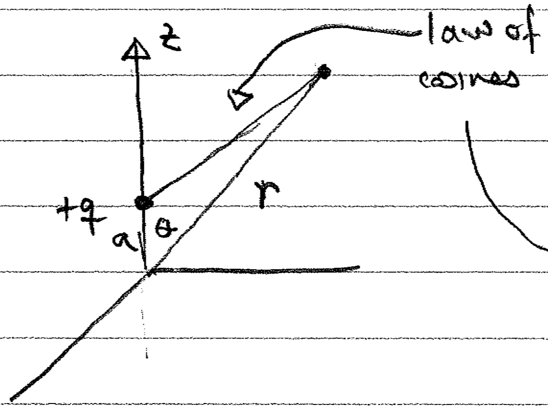
$$= 1 + xt - \tfrac{1}{2}t^2 + \tfrac{3}{8}(4x^2t^2 - 4xt^3 + t^4)$$
$$- \tfrac{5}{16}\left[-8x^3t^3 + 12x^2t^4 \dots\right]$$

$$= t^0\left[1\right] \qquad P_0(x) = 1$$
$$+ t^1\left[x\right] \qquad P_1(x) = x$$
$$+ t^2\left[-\tfrac{1}{2} + \tfrac{3}{2}x^2\right] \qquad P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2}$$
$$+ t^3\left[-\tfrac{3}{2}x + \tfrac{5}{2}x^3\right] \qquad P_3(x) = \tfrac{5}{2}x^3 - \tfrac{3}{2}x$$

-2//

Where does this generating function come from physically?

One way to see it: compute $V(\vec{r})$ for charge q a distance a from origin



$$V(r) = \frac{q}{4\pi\epsilon_0} \frac{1}{(r^2+a^2-2ar\cos\theta)^{1/2}}$$

$$V(r) = \frac{q}{4\pi\epsilon_0 r} \frac{1}{\left(1 - \frac{2a}{r}\cos\theta + a^2/r^2\right)^{1/2}}$$

Drop

Define $\cos\theta = x$ $\quad a/r = t$

$$(1 - 2tx + t^2)^{-1/2}$$ our generating function

$$V(r) = \frac{q}{4\pi\epsilon_0 r} \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n$$

Virtue: separates out angular structure from distance structure

As in QM $\psi_{n\ell m}(\vec{r}) = Y_{\ell m}(\theta,\phi)\, R_n(r)$

Legendre poly also arises from a particular differential Eqn, To see this

$$g(t,x) = (1-2xt+t^2)^{-1/2} = \sum_0^\infty P_n(x)\, t^n$$

$$\partial g/\partial x = \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_0^\infty P_n'(x)\, t^n$$

First

$$(1-2xt+t^2)\sum_0^\infty P_n'(x)\, t^n - t\sum_0^\infty P_n(x)\, t^n = 0$$

Coefficients of each power of t set to zero

$$P_n'(x) - 2x P_{n-1}'(x) + P_{n-2}'(x) - P_{n-1}(x) = 0$$

$$* \quad P_{n+1}'(x) + P_{n-1}'(x) = 2x\, P_n'(x) + P_n(x)$$

① Also
First

$$\partial g/\partial t = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_0^\infty n P_n(x)\, t^{n-1}$$

$$(1-2xt+t^2)\sum n P_n(x)\, t^{n-1} + (t-x)\sum P_n(x)\, t^n = 0$$

same trick of identifying powers of t:

$$(n+1) P_{n+1}(x) = (2n+1)\, x P_n(x) - n P_{n-1}(x)$$

$n=1$ $\quad 2P_2(x) = 3x\, P_1(x) - P_0(x) = 3x^2 - 1$ ✓

$n=2$ $\quad 3P_3(x) = 5x\, P_2(x) - 2P_1(x) = 5x(3x^2-1)\tfrac{1}{2} - 2x$
$\quad = 15x^3/2 - 9x/2$ ✓

This is called a "recussion relation",

Let's differentiate wrt x ; $\times 2$

$$(n+1)\, P'_{n+1}(x) = (2n+1)\, P_n(x) + (2n+1)\, x\, P'_n(x) - n\, P'_{n-1}(x)$$

$$\cancel{(2n+1) \times *} \quad \cancel{(2n+1)\, P'_{n+1}(x) + (2n+1)\, P'_{n-1}(x)}$$

$$\cancel{= 2x(2n+1)\, P_n(x) + (2n+1)\, P_n(x)}$$

This is a second eqn involving P'_{n+1}, P_n, P'_n, P_{n-1}

Playing around with combining this with $*$ one

can ultimately show

(Proof in text)

$$(1-x^2)\, P''_n(x) - 2x\, P'_n(x) + n(n+1)\, P_n(x) = 0$$

Price paid for getting eqn for just one $P_n(x)$

is that it is now <u>second</u> order

$$1-x^2 \frac{d^2}{dx^2} - 2x \frac{d}{dx} + (n+1)n$$

Exercise — Is this a Hermitian differential operator? If so $P_n(x)$ are <u>complete</u>

Talk about this more later in terms of our general theory of eigenfunctions of Hermitian operators

Put first ... special values

set $x = 1$ in generating funct.

$$\frac{1}{(1-2t+t^2)^{1/2}} = \frac{1}{1-t} = \sum_0^\infty t^n$$

But also $$= \sum P_n(1)\, t^n$$

so
$$\left.\begin{aligned} P_n(1) &= 1 \\ P_n(-1) &= (-1)^n \end{aligned}\right\} \text{ check it out}$$

orthogonality
$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \frac{2}{(2n+1)}\, \delta_{m,n}$$

Check cases ...

$$\int_{-1}^{1} P_1(x)\, P_3(x)\, dx = \int_{-1}^{1} x\left(\tfrac{5}{2}x^3 - \tfrac{3}{2}x\right) dx$$

$$= \int_{-1}^{1}\left(\tfrac{5}{2}x^4 - \tfrac{3}{2}x^2\right) dx = \tfrac{1}{2}x^5 - \tfrac{1}{2}x^3\Big|_{-1}^{1} = 0$$

$$\int_{-1}^{1} P_2(x)\, P_2(x)\, dx = \int_{-1}^{1}\left(\tfrac{3}{2}x^2 - \tfrac{1}{2}\right)^2 dx$$

$$= \frac{1}{4}\int_{-1}^{1}(9x^4 - 6x^2 + 1)\, dx = \frac{1}{4}\left(\tfrac{9}{5}x^5 - 2x^3 + x\right)_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{9}{5} - 2 + 1\right) = \frac{1}{2}\,\frac{4}{5} = \frac{2}{5} = \frac{2}{2(2)+1}$$

proofs of orthogonality

(2) Using generating function we get normalization

$$(1-2xt+t^2)^{-1} = \left(\sum_0^\infty P_n(x)t^n\right)^2$$

$$\int_{-1}^{1} dx\,(1-2xt+t^2)^{-1} = \sum_0^\infty t^{2n}\int_{-1}^{1}[P_n(x)]^2\,dx$$

$y = 1-2tx+t^2 \qquad dy = -2t\,dx$

$$\to \int_{(1+t)^2}^{(1-t)^2} \frac{dy/(-2t)}{y} = \frac{1}{2t}\ln y\Big|_{(1-t)^2}^{(1+t)^2} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right)$$

$\ln(1+$ [illegible] $- u^2/$ [illegible] $+ u^4/$ [illegible]

$\ln(1-u) =$ [illegible] $/2 -$ [illegible] $- u^4/4$

(1) Using general theory from 204A Sturm-Liouville theory

$$\mathcal{L} = p_0(x)\,d^2/dx^2 + p_1(x)\,d/dx + p_2(x)$$

$\mathcal{L}$ is self-adjoint (aka Hermitian) if $p_1(x) = p_0'(x)$

$$\left[(1-x^2)\,d^2/dx^2 - 2x\,d/dx + n(n+1)\right]P_n(x) = 0$$

✓

$\therefore$ $\{P_n(x)\}$ are orthogonal

Interval? $$\int_a^b f\mathcal{L}g\,dx = \int_a^b g\mathcal{L}f\,dx + p_0(x)\left[gf' - g'f\right]\Big|_a^b$$

$p_0(a) = p_0(b) = 0 \qquad a = -1,\ b = 1$

$$\ln(1+t) = t - t^2/2 + t^3/3 - t^4/4$$

$$\ln(1-t) = -t - t^2/2 - t^3/3 - t^4/4$$

$$\ln(1+t) = \int \frac{dt}{1+t} = \int (1 - t + t^2/2 - t^3/3 \ldots)\,dt$$

$$\ln[(1+t)/(1-t)] = \ln(1+t) - \ln(1-t)$$

$$= 2\left(t + t^3/3 + t^5/5 + \ldots\right)$$

$$\frac{1}{t}\ln\frac{1+t}{1-t} = 2\left(1 + \frac{t^2}{3} + \frac{t^4}{5} + \frac{t^6}{7}, \ldots\right) = \sum_0^\infty \frac{2}{2n+1}\,t^{2n}$$

$$= \sum_0^\infty \left[\int_{-1}^{1} [P_n(x)]^2\,dx\right] t^{2n}$$

$$\therefore \quad \int_{-1}^{1} (P_n(x))^2\,dx = \frac{2}{2n+1}$$

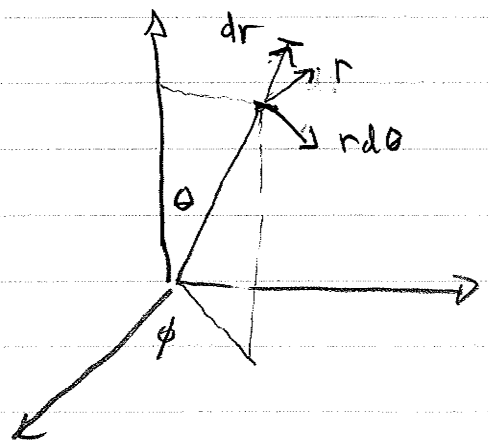
Applications of Legendre Poly's to EM

We have seen Coulomb's Law → Legendre polys from viewpoint of generating functions.

What about diff. eqn? $\nabla^2 V = 0$ if $\rho = 0$

What is ∇^2 in r, θ, ϕ coordinates

$$\vec{\nabla} V = \hat{r} 1 \frac{\partial V}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial V}{\partial \theta} + \hat{\phi} \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi}$$



$ds = 1dr$ if r changed $r + dr$

$ds = r\,d\theta$ if θ changed $\theta + d\theta$

$ds = r \sin\theta\, d\phi$ if ϕ changed $\phi + d\phi$

General rule: if, curvilinear coordinates $q_1\ q_2\ q_3$ used instead of $x\ y\ z$

But these coordinates are still orthogonal (see above)

$$\nabla^2 V = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \frac{h_2 h_3}{h_1} \frac{\partial V}{\partial q_1} + \frac{\partial}{\partial q_2} \frac{h_1 h_3}{h_2} \frac{\partial V}{\partial q_2} + \frac{\partial}{\partial q_3} \frac{h_1 h_2}{h_3} \frac{\partial V}{\partial q_3} \right]$$

~~Complex formula because~~ $\cancel{\hat{\theta}, \hat{\phi}, \hat{r}}$ ~~depend on~~

where $ds = \left. \begin{array}{l} h_1\, dq_1 \\ h_2\, dq_2 \\ h_3\, dq_3 \end{array} \right\}$ changing q_1, q_2, q_3 respectively

$r \quad h_1 = 1$

$\theta \quad h_2 = r$

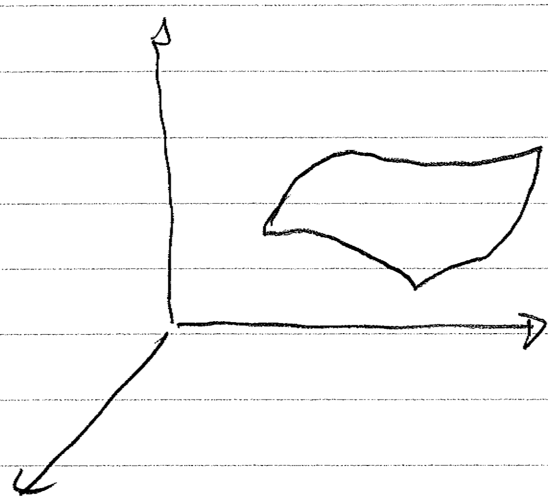
$\phi \quad h_3 = r\sin\theta$

$$\nabla^2 V = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\, r^2\sin\theta \frac{\partial V}{\partial r} + \frac{\partial}{\partial\theta}\,\sin\theta\frac{\partial V}{\partial\theta} + \frac{\partial}{\partial\phi}\,\frac{1}{\sin\theta}\frac{\partial V}{\partial\phi}\right]$$

why so complicated? $\hat{r}, \hat{\theta}, \hat{\phi}$ depend on r, θ, ϕ

as oppose to $\hat{x}\ \hat{y}\ \hat{z}$ being same everywhere in space

Differential geometry



← "Manifold" a lower dimensional object embedded in a higher d space

How can you define unit vectors, derivatives on manifold $\boxed{\text{w/o}}$ reference to the higher d space?!

L-10

$V(r,\theta,\phi) \to V(r,\theta)$ $\quad \nabla^2 V = 0$ but be a bit more general

$$V(r\,\theta\,\phi) = R(r)\,T(\theta)\,S(\phi)$$

$$\frac{1}{r^2}\, TS \frac{d}{dr} r^2 \frac{dR}{dr} + \frac{RS}{r^2\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dT}{d\theta} + \frac{RT}{r^2\sin^2\theta}\frac{d^2S}{d\phi^2}$$

$$= -k^2\, RTS$$

Helmholtz eqn

$$\frac{1}{S}\frac{d^2S}{d\phi^2} = r^2\sin^2\theta\left[-k^2 - \frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{1}{r^2\sin\theta\, T}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)\right]$$

rhs is indep of ϕ

m must be an integer?

$$\frac{1}{S}\frac{d^2S}{d\phi^2} = -m^2 \qquad S(\phi) = e^{\pm im\phi}$$

or $\cos m\phi \quad \sin m\phi$

$$\frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{r^2\sin\theta\, T}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) - \frac{m^2}{r^2\sin^2\theta} = -k^2$$

$$Q = \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + r^2k^2 = \frac{-1}{\sin\theta\, T}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) + \frac{m^2}{\sin^2\theta}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2R - \frac{QR}{r^2} = 0$$

$$+\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) - \frac{m^2}{\sin^2\theta}T + QT = 0$$

L-10A

$$D \nabla^2 \psi(r,t) = \frac{\partial \psi(rt)}{\partial t}$$

$$\psi = f(\bar{r})\, g(t)$$

$$D g \nabla^2 f = f \, {\partial g}/{\partial t}$$

$$\frac{\nabla^2 f}{f} = \frac{1}{Dg} \, {\partial g}/{\partial t} = -k^2$$

$$g = e^{-Dk^2 t}$$

$$\nabla^2 f = -k^2 f \qquad e^{ikx}$$

1-d:

$$\psi(xt) = \int_{-\infty}^{\infty} a(k)\, e^{ikx}\, e^{-Dk^2 t}$$

$\psi(x,0)$ determines $a(k)$ etc.