Permutations of 3 objects S_3

$I(abc) = abc$ How many elements? 6 orders $\Rightarrow$ 6 elements

$G_{12}(abc) = bac$

$G_{13}(abc) = cba$

$G_{23}(abc) = acb$

$G_{132}(abc) = bca$

$G_{312}(abc) = cab$

$G_{12}^{-1} = G_{12}$ $G_{12}G_{13}(abc) = G_{12}(cba) = bca = G_{132}(abc)$

	I	G_{12}	G_{13}	G_{23}	G_{132}	G_{312}
1	1	G_{12}	G_{13}	G_{23}	G_{132}	G_{312}
G_{12}	G_{12}	1	G_{132}	G_{312}	G_{13}	G_{23}
G_{13}			1	G_{132}	G_{23}	G_{12}
G_{23}				1	G_{12}	G_{13}
G_{132}				G_{13}	G_{312}	1
G_{312}				G_{12}	1	G_{132}

$G_{132}G_{312}(abc) = G_{132}(cab) = abc = I$

non abelian

$G_{12}G_{13}(abc) = G_{12}(cba) = bca = G_{132}$

$G_{13}G_{12}(abc) = G_{13}(bac) = cab = G_{312}$

Classes.

If $a, b, c \in G$ and $b = cac^{-1}$
Then b is conjugate to a

$$\overset{/\!/\ G_{13}}{}$$

Eg for S_3
$$G_{13}\, G_{23}\, G_{13}^{-1} (abc)$$
$$= G_{13}\, G_{23}\, (cba)$$
$$= G_{13} (cab)$$
$$= bac = G_{12}$$

So G_{23} and G_{12} are conjugate

1) a is always conjugate to itself
Proof: choose $c = I$ then $cac^{-1} = a$

2) If a conjugate to b, b conjugate to a

$$b = cac^{-1} \rightarrow c^{-1}bc = a$$
$$fbf^{-1} = a \quad \text{with } f = c^{-1}$$

3) If a conjugate to b and c then b and c are conjugate to each other

$\boxed{\#\text{classes} = \#\ \text{Irreducable representations}}$

$$a = fbf^{-1} \qquad a = gcg^{-1}$$
$$fbf^{-1} = gcg^{-1}$$
$$b = f^{-1}gcg^{-1}f = f^{-1}g\,c\,(f^{-1}g)^{-1}$$

Why is $(f^{-1}g)^{-1} = g^{-1}f$ check $f^{-1}gg^{-1}f = 1$.

G26

Compute all elements conjugate to G_{12}

$I G_{12} I^{-1} = G_{12}$

$G_{13} G_{12} G_{13}^{-1} = G_{23}$

$G_{23} G_{12} G_{23}^{-1} = G_{13}$

$G_{231} G_{12} G_{231}^{-1} = G_{23}$

$G_{132} G_{12} G_{132}^{-1} = G_{13}$

Set of elements conjugate to each other: Equivalence Class.

Same as subgroup?

No $G_{12} G_{13} = G_{132}$ so equivalence class is not closed under multiplication.

Are there subgroups of S_3 you can see?

$\{I, G_{12}\}$

Set of all integers under addition is a group

$n \text{ "x" } m \equiv n + m$

① closed ✓
② associative ✓
③ I ✓ $I = 0$!!
④ a^{-1} ✓ $a^{-1} = -n$

Why doesn't usual X work

$4 \times 3 = 12$ ① ✓
$4 \times (3 \times 2) = (4 \times 3) \times 2$ ② ✓
$4 \times 1 = 4$ ③ ✓
$4 \times ? = 1$ ④ ✗

G27

Subgroup ≡ Even integers !

Suppose G has a subgroup S and $a \in G$ but $a \notin S$

$\{a s_i\}$ where $s_i \in S$ ⇒ left coset of subgroup S with respect to a

$\{s_i a\}$ ⇒ right "

$a s_i$ and $s_i a$ cannot be in S if a is not

$a s_i \in S \qquad s_i \in S \Rightarrow s_i^{-1} \in S$

$a s_i s_i^{-1} = a \in S \quad \times$.

The set of elements conjugate to $s_i \in S$ form a subgroup S'

Closure ① $(a s_1 a^{-1})(a s_2 a^{-1}) = a \overbrace{s_1 s_2}^{\in S} a^{-1} \in S'$

Association ② $(a s_1 a^{-1} a s_2 a^{-1}) a s_3 a^{-1} = a s_1 a^{-1}(a s_2 a^{-1} a s_3 a^{-1})$

Identity ③ $I \in S$ so $a I a^{-1} = I \in S'$

④ $(a s a^{-1})^{-1} = a s^{-1} a^{-1}$

since $a s a^{-1} a s^{-1} a^{-1} = a s s^{-1} a^{-1} = a a^{-1} = I$

If $S' = S$ S is self-conjugate.

The collection of even integers is self conjugate or invariant

$$\begin{array}{cccc} a & \text{"x"} & s & a^{-1} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ n & + & 2m & + (-n) = 2m \end{array}$$

g_{28}.

$S = \{1, g_{12}\}$ is not self conjugate

$$g_{13}\, g_{12}\, g_{13}^{-1} = g_{23} \notin S$$

S_3 three classes $\Rightarrow$ three irreps.

one is the trivial rep $g_i \Rightarrow 1$ dim 1

two more $n_2^2 + n_3^2 = 5 \Rightarrow n_2 = 1 \quad n_3 = 2$

$$\begin{aligned} 1 &\rightarrow 1 \\ g_{12} &\rightarrow -1 \\ g_{13} &\rightarrow -1 \\ g_{23} &\rightarrow -1 \\ g_{132} &\rightarrow 1 \\ g_{312} &\rightarrow 1 \end{aligned}$$

Exchange

2×2 matrices which when squared give identity (because they are inverses of each other)

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad g_{12} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$g_{13} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ +\sqrt{3}/2 & +1/2 \end{pmatrix} \qquad g_{23} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$$

0

1 0 1

not unique

4 eqns in 4 unknowns

$b(a+d) = 0$

$c(a+d) = 0$

$b = 0 = c$

or $a + d = 0$

$a^2 + bc = 1$

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

$a = 1$

$$\begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$