The rotation group, angular momentum generators

unitary matrices $UU^+ = I$

$$U = e^{i\theta S} \qquad U^+ = e^{-i\theta S^+}$$

$$UU^+ = e^{i\theta(S-S^+)} = I \qquad S = S^+ \qquad S \text{ Hermitian}$$

$$\det U = e^{\operatorname{tr}\ln U} = e^{\operatorname{tr} i\theta S} = e^{i\theta \operatorname{tr} S} = 1 \qquad \operatorname{tr} S = 0$$

1 dim representation $\quad S = 0 \quad e^{i\theta S} = 1$

2 dim rep
$$\begin{pmatrix} x_{11} & x_{21} - iy_{21} \\ x_{21} + iy_{12} & -x_{11} \end{pmatrix}$$

$$= x_{11}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z} + x_{21}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x} + y_{12}\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_y}$$

$\sigma_x \ \sigma_y \ \sigma_z$ generate 2-dim rep of rotation group

$$[\sigma_x, \sigma_y] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 2i\,\sigma_z$$

Structure constant C^z_{xy} (C^3_{12})

$C^x_{xy} = C^y_{xy} = 0$

Also useful to write

$$\sigma_+ = (\sigma_x + i\sigma_y)\tfrac{1}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_- = (\sigma_x - i\sigma_y)\tfrac{1}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Whereas $$[\sigma_z, \sigma_x] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2i\sigma_y$$

$$[\sigma_z, \sigma_y] = -2i\sigma_x$$

Here $$[\sigma_z, \sigma_+] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2\sigma_+$$

$$[\sigma_z, \sigma_-] = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} = -2\sigma_-$$

Why interesting?

$$\sigma_z\sigma_+ - \sigma_+\sigma_z = 2\sigma_+$$

Suppose you have a vector which is an eigenstate of σ_z

$$\sigma_z |m\rangle = m|m\rangle$$

$$\sigma_z\sigma_+|m\rangle - \sigma_+\sigma_z|m\rangle = 2\sigma_+|m\rangle$$

$$\sigma_z\sigma_+|m\rangle = (2+m)\,\sigma_+|m\rangle$$

$\sigma_+|m\rangle$ is also an eigenstate of σ_z with eigenvalue $2+\lambda$!

You have seen this before in QM

$\frac{\hbar}{2}\vec{\sigma}$ spin ½ (angular momentum ½) matrices

$$\sigma_z|-\rangle = \underbrace{-\tfrac{1}{2}\hbar}_{m}|-\rangle$$

$\sigma_+|-\rangle$ also eigenstate of σ_z eigenvalue

$$\underbrace{-\tfrac{1}{2}\hbar}_{m} \;\underset{+\,2(\hbar/2)}{+\hbar} = +\hbar/2.$$

G22A $\longrightarrow \longrightarrow$

What about more complicated groups? SU(3) has eight generators HM 4.2.1. Two are diagonal. $H_1\ H_2$ $E_1\ E_2 \dots E_6$

Can you compute combination such that

$$[H_1, E_1] = \alpha_1 E_1 \qquad [H_1, E_2] = \alpha_1' E_2$$
$$[H_2, E_1] = \alpha_2 E_1 \qquad [H_2, E_2] = \alpha_2' E_2$$

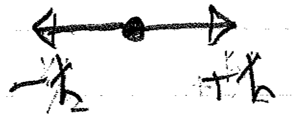
$$\vec{\alpha} = (\alpha_1, \alpha_2) \qquad \vec{\alpha}' = (\alpha_1' \alpha_2')$$

$$H_1|m_1 m_2\rangle = m_1|m_1 m_2\rangle$$
$$H_2|m_1 m_2\rangle = m_2|m_1 m_2\rangle$$

$$H_1 E_1|m_1 m_2\rangle = E_1 H_1|m_1 m_2\rangle + \alpha_1 E_1|m_1 m_2\rangle$$
$$= (m_1+\alpha_1)E_1|m_1 m_2\rangle$$
$$H_2 E_1|m_1 m_2\rangle = E_1 H_2|m_1 m_2\rangle + \alpha_2 E_1|m_1 m_2\rangle$$
$$= (m_2+\alpha_2)E_1|m_1 m_2\rangle$$

G22A

Root diagram



$$[\sigma_z, \sigma_+] = +2\sigma_+$$

$$\left[\frac{\hbar}{2}\sigma_z, \frac{\hbar}{2}\sigma_+\right] = \boxed{\hbar}\left(\frac{\hbar}{2}\sigma_+\right)$$

$$\left[\frac{\hbar}{2}\sigma_z, \frac{\hbar}{2}\sigma_-\right] = \boxed{-\hbar}\left(\frac{\hbar}{2}\sigma_-\right)$$

If $[\hat{H}_{at}, \hat{J}^2] = [\hat{H}_{at}, \hat{J}_z] = 0$

Eigenvalues of $\hat{H}_{at}$ in degenerate sets.

$\hat{H}_{at}$, $\hat{J}^2$ all same

$\hat{J}_z$ shifted by $\underline{\underline{\hbar}}$

Further restriction on J^2, J_z worked out in QM.

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$$[\hat{H}_{at}, \hat{J}^2] = [\hat{H}_{at}, \hat{J}_z] = 0 \qquad [\hat{J}^2, \hat{J}_z] = 0$$

$$\hat{H}_{at} = \vec{p}^{\,2}/2m - e^2/r \qquad \psi_{n,\ell,m}$$

Eigenvalues $\quad -\dfrac{13.6}{n^2} \qquad n = 1, 2, \ldots$ independent of $\underline{\underline{\ell, m}}$

solns to schroedinger eqn come in degenerate sets.

$$\psi_n(\ell, m)$$

Explains periodic table!

H	He	$n=1 \quad \ell=0 \quad m=0$
Li Be B C N O F	Ne	$n=2 \quad \ell=1 \quad m=-1,0,1$ $\ell=0 \quad m=0$
etc		

QCD $\qquad n \quad p \quad$ known

$\Sigma^- \quad \Sigma^0 \Lambda \quad \Sigma^+$
$\Xi^- \quad \Xi^0$ $\Big\}$ discovered 1960's accelerators.

8 degenerate solns to $\hat{H}_{QCD}$

What could give this?

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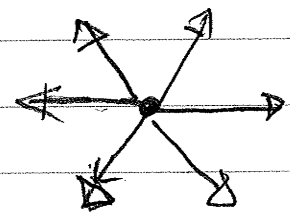
original eigenvalues (m_1, m_2) shifted to $(m_1 + \alpha_1, m_2 + \alpha_2)$

If you do this for SU(3) Exercise $\boxed{4.3.2}$

you find the shifts (α_1, α_2) look like



"root diagram"

$$[\hat{H}_{QCD}, \hat{H}_1] = 0$$
$$[\hat{H}_{QCD}, \hat{H}_2] = 0 \quad \Rightarrow \text{ 8 eigenstates of } H_{QCD} \text{ all with same energy}$$

2 diagonal generators of SU(3)