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$$M: \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ +\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

represents the order 3 group with multiplication table:

	1	a	b
1	1	a	b
a	a	b	1
b	b	1	a

Clearly $M = A \oplus D$ $\quad D = 1, 1, 1$

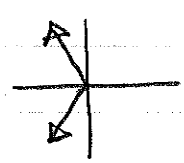
the trivial representation of the group

Is this reducible?

Is there an S which will diagonalize all the A_i $\quad S A_i S^{-1} = D_i$

$$\left(-\tfrac{1}{2} - \lambda\right)^2 + \tfrac{3}{4} = 0 \qquad \text{Same characteristic Eqn}$$

$$-\tfrac{1}{2} - \lambda = \pm \tfrac{\sqrt{3}}{2} i \qquad \lambda = -\tfrac{1}{2} \mp \tfrac{\sqrt{3}}{2} i = \begin{matrix} e^{2\pi i/3} \\ e^{4\pi i/3} \end{matrix}$$



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Eigenvectors

$\lambda = -1/2 - \sqrt{3}/2\, i$

M_2: $\sqrt{3}/2 \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$ $\quad \begin{pmatrix} 1 \\ -i \end{pmatrix}$

M_3: $\sqrt{3}/2 \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$ $\quad \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\lambda = -1/2 + \sqrt{3}/2\, i$

M_2: $\frac{\sqrt{3}}{2} \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}$ $\quad \begin{pmatrix} 1 \\ i \end{pmatrix}$

M_3: $\frac{\sqrt{3}}{2} \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$ $\quad \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Same eigenvectors!

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$ Diagonalizes <u>both</u>

Another way $\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

so they commute, $\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{4\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \end{pmatrix}$$

Irreducible finally! Notice the same representation appears twice. That can happen

Usually reps have different dimension

Theorem: Group of order m.

You have a representation. You reduce it until irreducible as we did for 3x3 matrices of C_3.
Suppose resulting block matrices are dimension n_1 n_2 ..

$$n_1^2 + n_2^2 + \ldots \leq m$$

For C_3 $n_1 = n_2 = n_3 = 1$ and $m = 3$.

* The <u>character</u> of a representation is the set of traces of matrices representing the group elements

character of C_3 rep $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

is $\{3, 0, 0\}$

Also $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

If two reps have matrices related by $M' = SMS^{-1}$ they are said to be equivalent.

characters same $\Leftrightarrow$ equivalent

Generators

We have considered mostly finite groups,
Let's start thinking about ∞ order groups and how we parameterize them

$SO(2)$ rotations in $d=2$ $\qquad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R(\theta)$

$$R(\theta) = \cos\theta \, I + \sin\theta \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{i\sigma_2} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

θ small $\qquad = I + i\sigma_2\theta$

$$\approx e^{i\sigma_2\theta} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In fact $\quad R(\theta) = e^{i\sigma_2\theta}$ for all θ not just small ones

σ_2 is called the generator of $SO(2)$

$R = e^{i\theta S}$

$$I = RR^{\dagger} = e^{i\theta S} e^{i\theta S^{\dagger}} = e^{i\theta(S+S^{\dagger})}$$

S must be <u>Hermitian</u>

and σ_2 is indeed.

615. Rotate about $\quad X \quad Y \quad Z$

SO(3)

$$S_1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad S_2 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad S_3 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Any rotation can be written $R = e^{i\vec{\omega}\cdot\vec{S}}$

$$\vec{\omega}\cdot\vec{S} = i\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$ Hermitian

$S_1 S_2 - S_2 S_1 = i S_3$ generators do not commute
neither do rotations
non Abelian!

$$S_1 S_2 = -\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_1 S_2 - S_2 S_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_2 S_1 = -\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$= i S_3$

$$[S_i, S_j] = \sum_k c_{ij}^{\ k} S_k$$

$\equiv$ structure constants of G

Less trivial example: Irreducible reps of character of ...

The permutation group S_3

1	$abc \to abc$	order (6 possible rearrangements)
g_{12}	bac	
g_{13}	cba	
g_{23}	acb	

$g_{23}\, g_{13}\,(abc) = g_{23}\,(cba) = cab$ not on list already

call it g_{312}

$g_{132}\,(abc) = bca$

	1	g_{12}	g_{13}	g_{23}	g_{132}	g_{312}
1	1	g_{12}	g_{13}	g_{23}	g_{132}	g_{312}
g_{12}		1	g_{132}			
g_{13}			1			
g_{23}			g_{312}	1		
g_{123}					1	
g_{321}						1

$g_{12}\, g_{13}\,(abc) = g_{12}(cba) = bca$