M 3/20 @ 2pm

A representation of an abstract group G is a homomorphism $f: G \to M$ where M is a group of $n \times n$ matrices. The representation is said to have dimension n. If the mapping is an isomorphism, the representation is said to be faithful.

* Every group has a trivial $n=1$ dimensional representation where each member of the group is represented by the number 1. Preserves group operation but we have lost a lot of info about group! The representation is not faithful!

Another representation of dimension equal to the order (# elements) of the group is this

This is called the "regular representation"

$g_1, g_2, \ldots, g_n$

If $g_i g_j = g_k$

Then $(M_i)_{j\ell} = \delta_{\ell k}$!

$$\cancel{(M_i M_j)_{st} = (M_i)_{sp}(M_j)_{pt}}$$

$$\cancel{= \delta}$$

$$g_i \, g_j = g_k$$

Want $(M_i)(M_j) = M_k$

$$(M_i M_j)_{pq} = (M_k)_{pq}$$

$$(M_i)_{pa}(M_j)_{aq} = (M_k)_{pq}$$

Q8a

	1	a	b
1	1	a	b
a	a	b	1
b	b	1	a

In words "if $g_i g_j = g_k$ Then in row j of matrix M_i put a 1 in column k.

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad M_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

first row ok obviously

$$M_2 M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = M_3 \quad ✓$$

$$M_2 M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = M_1 \quad ✓ \quad \text{etc.}$$

$$g_1 g_1 = g_1 \qquad g_1 g_2 = g_2 \qquad g_1 g_3 = g_3$$

$$(M_1)_{1\ell} = \delta_{\ell 1} \qquad (M_1)_{2\ell} = \delta_{\ell 2} \qquad (M_1)_{3\ell} = \delta_{\ell 3}$$

$$g_i g_j = g_k \qquad g_2 g_1 = g_2 \qquad g_2 g_2 = g_3 \qquad g_2 g_3 = g_1$$

$$(M_i)_{j\ell} = \delta_{\ell k} \qquad (M_2)_{1\ell} = \delta_{\ell 2} \qquad (M_2)_{2\ell} = \delta_{\ell 3} \qquad (M_2)_{3\ell} = \delta_{\ell 1}$$

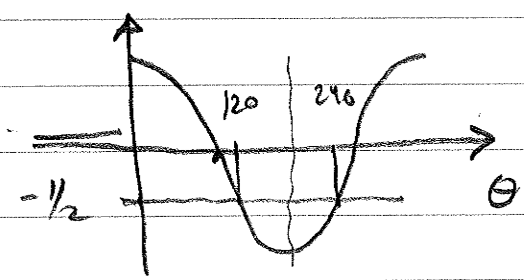
another 1d representation of order 3 group of page 68a besides $g_i \to 1$ is

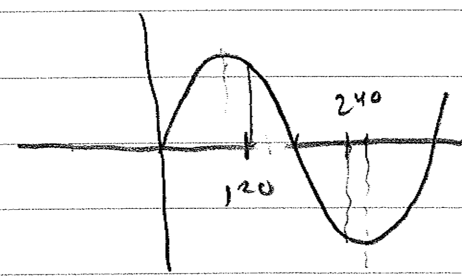
$$g_1 \Rightarrow 1$$
$$g_2 \to e^{2\pi i/3}$$
$$g_3 \to e^{4\pi i/3}$$

These complex #'s rotate by 120° in complex plane, suggesting a 2d rep of the group

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -1/2 & +\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \qquad \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ +\sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$\begin{matrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{matrix}$$





dimension 3 representation

3×3 Matrices representing our order 3 group

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Once you have a representation (matrices) you can consider vectors.

Notice that the 3×3 matrices above leave the third component of vector alone.

But the 3×3 matrices of page G8a (the "regular representation") do not.

In some sense G8c matrices are a "better" dimension 3 representation of the group because the existence of the invariance is more obvious.

* A representation in which the invariance of a subspace is obvious is called reducible

Another way: If g_i are represented by block matrices of the form

$$M_i = \begin{array}{c} \\ m \\ p \end{array}\!\!\begin{pmatrix} A_i & 0_i \\ B_i & D_i \end{pmatrix}$$

A_i $m \times m$
B_i $p \times m$
D_i $p \times p$

$n = m + p$ is dimension of representation

then the representation is reducible

G 10

$g_i g_j$

$$\begin{pmatrix} A_i & 0 \\ B_i & D_i \end{pmatrix}\begin{pmatrix} A_j & 0 \\ B_j & D_j \end{pmatrix} = \begin{pmatrix} A_i A_j & 0 \\ B_i A_j + D_i B_j & D_i D_j \end{pmatrix}$$

same form

$$\begin{pmatrix} A_i & 0 \\ B_i & D_i \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A_i u \\ B_i u + D_i v \end{pmatrix}$$

If $\begin{pmatrix} 0 \\ v \end{pmatrix}$ get $\begin{pmatrix} 0 \\ D_i v \end{pmatrix}$

this subspace is invariant.

If $B_i = 0$ also then $\begin{pmatrix} u \\ 0 \end{pmatrix}$ gets $\begin{pmatrix} A_i u \\ 0 \end{pmatrix}$

and upper subspace also invariant.

"<u>completely reducible</u>"

We write $M = A \oplus D$

not a usual sum. ~

but