<u>Classical Mechanics</u> : symmetry → conservation laws

How to say this? $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial p_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$ $\quad \mathcal{L}(p_1 p_2 \ldots q_1 q_2)$

$$\Downarrow$$

$$0 \Rightarrow \frac{\partial \mathcal{L}}{\partial p_i} = \text{const}$$

<u>Quantum mechanics</u> $\frac{\partial \hat{A}}{\partial t} = i\hbar [\hat{H}, \hat{A}]$ (Heisenberg picture)

$$\hat{H} = \hat{p}^2/2m + V(\cancel{x}) \to 0$$

$$[\hat{H}, \hat{p}] = 0 \qquad \hat{p}(t) = \hat{p}(0)$$

$$\langle \hat{p} \rangle = \text{const}$$

alternatively $[\hat{H}, \hat{A}] = 0 \to \hat{H}, \hat{A}$ can be diagonalised simultaneously

→ if $|\psi\rangle$ is eigenstate of $\hat{H}$ also eigenstate of $\hat{A}$

<u>Group theory</u> : A formal mathematical way to treat symmetries

Discrete groups symmetry occurs for finite # of elements (angles of)
Continuous <u>Lie</u> groups " " " infinite # of elements

Group

set of objects which can be combined with operator $*$

(1) $a, b \in G \rightarrow a * b \in G$ "G is closed"

(2) $a, b, c \in G \rightarrow (a*b)*c = a*(b*c)$

(3) $\exists I \ \forall a \in G \rightarrow a * I = I * a = a$

I is unique

(4) $\forall a \ \exists a^{-1} \in G \quad a * a^{-1} = a^{-1} * a = I$

a^{-1} is unique

$*$ subset $G' < G$ is closed G' is a subgroup

$\{I\}$ is a subgroup of G

$*$ $g \in G \quad g' \in G' \quad g g' g^{-1} \in G' \rightarrow G'$ is an invariant subgroup of G

$\{I\}$ is an invariant subgroup of G

similarity transformations

$*$ $a * b = b * a$ abelian group

order of group = # elements (can be infinite!)

<u>order 1</u> $\{I\}$ only 1 group

<u>order 2</u> $\{I, g\}$ $g = g^{-1}$

only 1 group

	I	g
I	I	g
g	g	I

reflection in plane or rotation by π around axis

<u>order 3</u> $\{I, g, h\}$

prove only 1!

	I	g	h
I	I	g	h
g	g	h	I
h	h	I	g

<u>permutation group</u> of collection of n objects

$I\,(a\ b\ c) \Rightarrow a\ b\ c$ $\quad n=3$

$g_{13}\,(a\ b\ c) \Rightarrow (c\ b\ a)$

$g_{23}\,(a\ b\ c) \Rightarrow (a\ c\ b)$

How many elements?

Is it Abelian?

G3

Examples $\{$orthogonal $n \times n$ matrices$\}$ $= O(n)$

$SO(n)$ also $\det = 1$

$\det O \det O^T = 1$

~~$so(n)$~~ orthogonal $\quad O O^T = I \quad \det O = \det O^T$

(1) $(O_1 * O_2) \in G$ if $O_1, O_2 \in G$?

$$(O_1 * O_2) * (O_1 * O_2)^T = O_1 * O_2 * O_2^T * O_1^T \quad \text{use associative}$$
$$= O_1 * O_1^T$$
$$= I \quad \checkmark$$

(2) matrix multiplication is associative

$$[(O_1 * O_2) * O_3]_{ij} = (O_1 * O_2)_{ik} (O_3)_{kj}$$
$$= (O_1)_{in} (O_2)_{nk} (O_3)_{kj}$$
$$= (O_1)_{in} (O_2 * O_3)_{nj} = [O_1 * (O_2 \times O_3)]_{ij}$$

(3) $I = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix} \qquad O_1 * I = I * O_1 = O_1$

(4) $(O_1)^{-1} = O_1^T$

6.4

orthogonal $O O^T = D$ ← diagonal

(1) $(O_1 * O_2) * (O_1 * O_2)^T = O_1 * O_2 * O_2^T * O_1^T$

$= O_1 * D_2 * O_1^T$

diagonal matrices commute with all other matrices!

$= O_1 * O_1^T * D_2$

$= D_1 * D_2 = D$ ✓✓

(2) ✓ Associative

(3) ✓ I

(4) $O^{-1} = O^T D^{-1} = D^{-1} O^T$

$$(O^{-1})(O^{-1})^T = (O^T D^{-1})(O^T D^{-1})^T$$
$$= O^T D^{-1} D^{-1} O$$
$$= D^{-2} O^T O = D^{-2} D = D^{-1}$$

How many? $n=2$ $\begin{array}{cc} \vec{v}_1 & \vec{v}_2 \\ \end{array} \begin{pmatrix} \downarrow & \downarrow \end{pmatrix}$ $v_2 \perp v_1$ is unique except for length

If v_1 normalized only 1 param

only 1 free parameter $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

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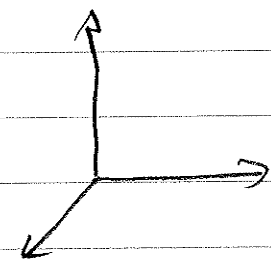
How many $\begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \multicolumn{3}{c}{\left(\begin{array}{ccc} \quad & \quad & \quad \\ & & \end{array} \right)} \end{array}$



2 parameters if $|\vec{v}_1| = 1$

$\vec{v}_2$ is in plane $\perp$ to $\vec{v}_1$ $\rightarrow$ 1 parameter

$\vec{v}_3$ unique

3

General rule $n(n-1)/2$

$\vec{v}_1$ $n-1$

$\vec{v}_2$ $n-2$

$\vdots$

v_n unique

$$\sum_{i=1}^{n-1} i = \frac{(n-1)(n-1+1)}{2}$$

Unitary matrices $UU^{\dagger} = I$ $SU(n)$

similar arguments ...

$SO(n)$, $SU(n)$ are Lie groups (continuous parameters)

They are also "compact" parameters are on finite intervals.

Correspondence between elements of G_1 and G_2'

preserving multiplication : homomorphic

$$g_1, g_2 \in G$$
$$\Downarrow$$
$$g_1', g_2' \in G'$$
$$(g_1 * g_2)' = g_1' * g_2'$$

$SO(3)$ is homomorphic to $SU(2)$!

If 1-1 correspondence isomorphic

If G homomorphic to group of matrices G'

then G' is a representation of G

If G, G' are isomorphic faithful representation

* Representations are not unique !

Symmetries are understood nicely in terms of matrix representation

Symmetry means $RHR^{-1} = H$ for $R \in G$

$\Leftarrow$ $RH = HR$ $[H, R] = 0$

then if $H\psi = E\psi$

$$HR\psi = RHR^{-1}R\psi = RH\psi = RE\psi = ER\psi$$

* $R\psi$ is also eigenvector with same eigenvalue
"multiplet"

$$H = p^2/2m - e^2/r$$

H commutes with L^2 and L_z
rotation matrices (also spin)

Eigenvalues E_n are degenerate $E_{n,l,m}$

$l = 0, 1, \ldots n-1$
$m = -l, \ldots l$

$n = 1$ $l = m = 0$ H, He

$n = 2$ $l = 0$ $m = 0$
$l = 1$ $m = -1, 0, 1$ } Li Be B C N O F Ne

A representation of G is irreducible if the use of all elements of G give the entire set of vectors which have the degenerate eigenvalue.

2s and 2p