Completeness + Orthogonality of Bessel Functions

$$\left[x^2 \frac{d^2}{dx^2} + x\frac{d}{dx} + (k^2x^2 - n^2)\right] J_n(kx) = 0$$

Recall Sturm-Liouville theory

$$\mathcal{L} \equiv p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$$

is Hermitian if $p_1(x) = p_0'(x)$

In such a situation the eigenfunctions $f_\lambda(x)$

$$\mathcal{L} f_\lambda(x) = \lambda f_\lambda(x)$$

are a complete set:

$$g(x) = \int d\lambda\, c(\lambda)\, f_\lambda(x)$$ ← any g can be expanded in f_λ

and f_λ are also orthogonal:

$$\int dx\, f_\lambda(x) f_{\lambda'}(x) = \delta_{\lambda,\lambda'} \text{ or } \delta(\lambda-\lambda')$$

discrete λ continuous λ.

The Bessel differential operator is Hermitian only after dividing by the "weight function" $w(x) = x$

$$x \frac{d}{dx} + \frac{d}{dx} + \left(k^2 x^2 - n^2/x\right) \qquad J_n(kx) = 0$$

$p_0(x) = x$ $\quad$ $p_1(x) = 1$ $\qquad$ $p_1 = p_0'$

In this case, as we discussed, the eigenfunctions obey a generalized orthogonality

$$\int w(x) f_\lambda(x) f_{\lambda'}(x)\, dx = \delta_{\lambda\lambda'} \qquad \delta(\lambda - \lambda')$$

Let's work out how this occurs explicitly for Bessel functions. Consider problem where we require functions $(x \to \rho)$ to vanish at $\rho = a$. When we solved Schroedinger eqn we saw this quantized the K values

$$J_n(\alpha_{nm}\, \rho/a) \qquad \text{where } \alpha_{nm} \text{ are } m\text{th roots of } J_n: \; J_n(\alpha_{nm}) = 0$$

Aside: Orthogonality and completeness depends not only on $\mathcal{L}$ but also on boundary conditions $\frac{d^2}{dx^2} f = -k^2 f$ $\quad 0 < x < a \;\rightarrow\; \sin\frac{n\pi x}{a}$

$f(x) = 0$

$x = 0, a$

The derivation below follows very closely last quarter's derivation of the $\rho_1 = \rho_0$ Hermiticity condition

$\cdot J_n(\alpha_{nm'}\rho/a)$

$$\left[\rho \frac{\partial^2}{\partial\rho^2} J_n\left(\alpha_{nm}\frac{\rho}{a}\right) + \frac{\partial}{\partial\rho} J_n\left(\alpha_{nm}\frac{\rho}{a}\right) + \frac{\alpha_{nm}^2 \rho}{a^2} - \frac{\nu^2}{\rho}\right] J_n\left(\alpha_{nm}\frac{\rho}{a}\right) = 0$$

$$\left[\rho \frac{\partial^2}{\partial\rho^2} J\left(\alpha_{nm'}\frac{\rho}{a}\right) + \frac{\partial}{\partial\rho} J_n\left(\alpha_{nm'}\frac{\rho}{a}\right) + \frac{\alpha_{nm'}^2}{a^2}\rho - \frac{\nu^2}{\rho}\right] J_n\left(\alpha_{nm'}\frac{\rho}{a}\right) = 0$$

$\cdot J_n(\alpha_{nm}\rho/a)$

and subtract

$$J_n\left(\alpha_{nm'}\frac{\rho}{a}\right)\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho} J_n\left(\alpha_{nm}\frac{\rho}{a}\right)\right] - J_{nm}(\alpha_{nm}\rho/a)\frac{\partial}{\partial\rho}\left[\rho\,\partial/\partial\rho\, J_n(\alpha_{nm'}\rho/a)\right] = \frac{\alpha_{nm}^2 - \alpha_{nm'}^2}{a^2}\rho J_n\left(\alpha_{nm}\frac{\rho}{a}\right) J_n\left(\alpha_{nm'}\frac{\rho}{a}\right)$$

Integrate from 0 to a and then integrate by parts

The "integral term" vanishes because one has two identical pieces differing by a $(-)$ sign. The "surface term" vanishes at $\rho = \emptyset$ because of the ρ factor and at $\rho = a$ because of $\alpha_{nm}\rho/a$ argument.

Thus as long as $\alpha_{nm}^2 \neq \alpha_{nm'}^2$ we have

$$\int_0^a \rho J_n(\alpha_{nm}\rho/a) J_n(\alpha_{nm'}\rho/a)\, d\rho = 0$$

ORTHOGONALITY

Normalization $\displaystyle\int_0^a \left[J_n(\alpha_{nm}\rho/a)\right]^2 \rho\, d\rho = \frac{a^2}{2}\left[J_{n+1}(\alpha_{nm})\right]^2$

$\boxed{\text{Exercise}}$ from recurrence reln (HW 1-1)

$f(0) = 0$
$f(a) = 0$

Summary we can expand $f(\rho)$ $0 < \rho < a$

$$f(\rho) = \sum_{m=1}^{\infty} C_{nm} J_n(\alpha_{nm} \rho/a)$$

$$C_{nm} = \frac{2}{a^2 (J_{n+1}(\alpha_{nm}))^2} \int_0^a f(\rho) J_n(\alpha_{nm} \rho/a) \rho \, d\rho$$

Q: Why are $\{J_n(\alpha_{nm} \rho/a)\}$ complete for each n?

Shouldn't we have to include all $\underline{n}$ in expansion?

A: Different δ, PDE for each n