# Bessel Functions (Laplacian in cylindrical coordinates)

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + 1/\rho\, \partial/\partial\rho\, \rho\, \partial/\partial\rho$$

Diffusion Eqn $\quad D\nabla^2\psi = \frac{\partial \psi}{\partial t}$

Try soln of form $\quad \psi(\rho,\phi,z,t) = A(\rho)\,B(z)\,e^{im\phi}\,e^{-\alpha t}$

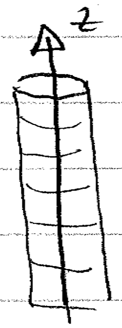
We are experienced enough to guess the $\phi, t$ dependence...

$$D\left[\frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial}{\partial\phi^2} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right] A B e^{im\phi} e^{-\alpha t} = -\alpha A B e^{im\phi} e^{-\alpha t}$$

$$\frac{1}{A}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{(\alpha - m^2)}{\rho^2}\right) A(\rho) + \frac{1}{B}\frac{\partial^2}{\partial z^2} B(z)$$

## Bessel Function (Laplacian in cylindrical Coordinates)

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}$$




∞ long cylinder + rotational symmetry abt z axis

We will first consider case of no $z$, $\phi$ dependence since Bessel functions already appear there. Then we will consider less symmetrical case

{ Schroedinger Eqn in "Pillbox" ← Class
{ Laplace Eqn in "pillbox" ← HW

$$D\nabla^2\psi = \partial\psi/\partial t \qquad \psi = A(\rho)\,e^{-\alpha t}$$

$$D\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\right)A\cancel{e^{-\alpha t}} = -\alpha A\cancel{e^{-\alpha t}}$$

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\alpha}{D}\right)A(\rho) = 0$$

$$\left[\rho^2\frac{\partial^2}{\partial \rho^2} + \rho\frac{\partial}{\partial \rho} + \left(\frac{\alpha}{D}\right)\rho^2\right]A(\rho) = 0$$

"$k^2$"

soln of $\nabla^2\psi = -k^2\psi$
also stat mech!

# Bessel Functions

Thus In solving diffusion Eqn for cylindrical geometry

we encounter

$$\left[x^2\frac{d}{dx^2} + x\frac{d}{dx} + k^2x^2\right]u(x) = 0$$

The solns are a special case $(n=0)$ of the Bessel Eqn

$$* \qquad \left[x^2\frac{d}{dx^2} + x\frac{d}{dx} + (x^2-n^2)\right]J_n(x) = 0$$

Ie $\quad u(x) = J_0(kx)$.

{ note that if $s \equiv kx$ $\quad s\frac{d}{ds} = x\frac{d}{dx}$ and $s^2\frac{d^2}{ds^2} = x^2\frac{d^2}{dx^2}$

so that $\left(x^2 d^2/dx^2 + x\, d/dx + k^2x^2\right)u(kx) =$

$$= \underbrace{\left(s^2\frac{d^2}{ds^2} + s\, d/ds + s^2\right)u(s)}_{\text{Bessel for } n=0}$$ }

What are Bessel functions, the solns to $*$ ?!

Basically they are like sines and cosines (oscillatory)

except they also decay, and zeroes not evenly spaced

B-2

Sines and cosines can be defined in different ways.

Most simply $\quad d^2/dx^2\, u(x) = -k^2 u$

But also $\left\{ \begin{array}{l} \sin x = x - x^3/3! + x^5/5! - \dots \\ \cos x = 1 - x^2/2! + x^4/4! - \frac{x^6}{6!} \end{array} \right.$

$\frac{d^2 u}{dx^2} = +k^2 u$ $\quad u = \sinh, \cosh$ series?

- $\sin, \cos$ complete $\Rightarrow$ Fourier rep
- $\frac{1}{\pi}\int_0^{2\pi} \sin nx \sin mx \, dx = \delta_{nm}$
- $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-k')x} dx = \delta(k-k')$

For Bessel functions

$$J_n(x) = \sum_s 1/s!(s+n)! \, (-1)^s \left(\frac{x}{2}\right)^{2s+n}$$

eg $$J_0(x) = \sum_s 1/(s!)^2 (-1)^s \left(\frac{x}{2}\right)^{2s}$$

$$= 1 - \frac{x^2}{4} + \frac{x^4}{4\cdot 16} - \frac{x^6}{36\cdot 64}$$

$$J_1(x) = \sum_s 1/s!(s+1)! \, (-1)^s \left(\frac{x}{2}\right)^{2s+1}$$

$$= \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{96} - \dots$$

Where does this series expansion come from?

$$\frac{d}{dx}\sin x = 1 - x^2/2! + x^4/4!$$

$$d^2/dx^2 \sin x = -x + x^3/3! = -\sin x$$

$$\frac{d}{dx} J_0(x) = -\frac{x}{2} + \frac{x^3}{16} - \frac{x^5}{6 \cdot 64}$$

$$\frac{d^2}{dx^2} J_0(x) = -\frac{1}{2} + \frac{3}{16} x^2 - \frac{5x^4}{6 \cdot 64}$$

$$x^2 \frac{d^2}{dx^2} J_0(x) + x \frac{d}{dx} J_0(x) + x^2 J_0(x)$$

$$= -\frac{1}{2} x^2 + \frac{3}{16} x^4 - \frac{5}{6 \cdot 64} x^6$$

$$-\frac{x^2}{2} + \frac{x^4}{16} - \frac{x^6}{6 \cdot 64}$$

$$+ x^2 - \frac{x^4}{4} + \frac{x^6}{4 \cdot 16} = 0$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$0 \quad 0 \quad 0$$

So (1) Bessel DE
(2) SERIES
(3) other...

~~So it works but ... where does it come from~~

$$e^{x/2(t - 1/t)} = \sum_n J_n(x) t^n$$

generating function for Bessel

Can show (1) gives series of page B-2

(2) satisfies Bessel DE of page B-1.

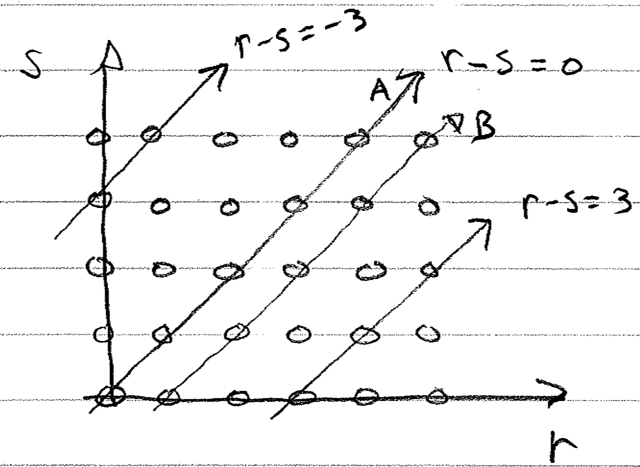
Basically a bunch of a tedious algebra...

HW: Use $g(x,t)$ to prove $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$

B-4

Proof of (1)

$$e^{xt/2}\, e^{-x/2t} = \sum_{r=0}^{\infty} \left(\frac{x}{2}\right)^r \frac{t^r}{r!} \sum_{s=0}^{\infty} \left(\frac{x}{2}\right)^s (-1)^s \frac{t^{-s}}{s!}$$




$t^{r-s}$

$n = r-s$

$s = r-n$

$r = s+n$

can relabel double sum using n and s instead

n can take any value $-\infty$ to $+\infty$

If $n \geq 0$ s will start at $\phi$

$$\left(\sum_{n=-\infty} \sum_{s=-n}^{\infty} + \right) \underbrace{\sum_{n=0}^{\infty} \sum_{s=0}^{\infty}} \left(\frac{x}{2}\right)^{s+n} \frac{t^{s+n}}{(s+n)!} \left(\frac{x}{2}\right)^s (-1)^s \frac{t^{-s}}{s!}$$

$n=0 \quad s=0,1,2,3\ldots \longrightarrow$ line A since $r=s+n=s$

$n=1 \quad s=0,1,2,3\ldots \longrightarrow$ line B same $r=s+n=s+1$

Focus on this to get $J_n$ for $n>0$

$$= \sum_{n=0}^{\infty} t^n \sum_{s=0}^{\infty} \frac{1}{s!} \frac{1}{(s+n)!} (-1)^s \left(\frac{x}{2}\right)^{2s+n}$$

Considering $n<0$ in same way leads to

$$J_n(x) = -J_{-n}(x)$$

B-5

$$g(x,t) = e^{x/2(t - 1/t)}$$

proof of (2)

$$d/dx \; g(x,t) = 1/2 (t - 1/t) e^{x/2(t - 1/t)}$$

$$d^2/dx^2 \; g(x,t) = 1/4 (t - 1/t)^2 e^{x/2(t - 1/t)}$$

$$\therefore \left(x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} + x^2 - n^2\right) g(x,t)$$

$$= \left[x^2 \frac{1}{4}\left(t - \frac{1}{t}\right)^2 + x \, \frac{1}{2}\left(t - \frac{1}{t}\right) + x^2 - n^2\right] e^{x/2(t - 1/t)}$$

$$= \left[x^2 \frac{1}{4} t^2 - \frac{x^2}{2} + \frac{x^2}{4t^2} + \frac{xt}{2} - \frac{x}{2t} + x^2 - n^2\right] \quad ''$$

, , ,

Further connections to familiar $\sin x$ and $\cos x$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \cdots \qquad \cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} \cdots$$

$$= \sum_s \frac{1}{(2s)!} x^{2s} (-1)^s \qquad = \sum_s \frac{1}{(2s)!} x^{2s}$$

$$\cos(ix) = \cosh x$$

similarly

$$J_n(x) = \sum \frac{1}{s!} \frac{1}{(s+n)!} \left(\frac{x}{2}\right)^{2s+n} (-1)^s$$

eliminated

$$I_n(x) =$$

"Modified Bessel Function"

others Neumann, Hankel are essentially like

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x, \sin x \quad \text{vs} \quad e^{ix}, e^{-ix}$$

B-7

An interesting identity comes from replacing t in generating function by $e^{i\theta}$

$$\frac{x}{2}(t - 1/t) \to x \cos\theta$$

$$e^{x\cos\theta} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\theta}$$

We will see next quarter that this allows us to solve a very interesting statistical mechanics problem, "the XY chain" in terms of Bessel functions.