Physics 204A, Fall 2010, Problem Set 7

[1.] A string is clamped at both ends $x = 0$ and $x = L$. The string is set in motion by a sharp blow at $x = a$, so that

$$y(x,0) = 0 \qquad\qquad \frac{\partial y(x,t)}{\partial t} = L\, v_0\, \delta(x-a)$$

at $t = 0$. The constant L is included in the velocity initial condition to compensate for the dimensions (inverse length) of $\delta(x-a)$. Solve the wave equation subject to these initial conditions.
Hint: It is useful to observe that,

$$\delta(x-a) = \frac{2}{L}\sum_{n=1}^{\infty} \sin(\frac{n\pi a}{L})\sin(\frac{n\pi x}{L})\ .$$

Can you convince yourself this is correct?

[2.] In class we introduced the generating function of the Bessel polynomials,

$$g(x,t) = \exp\Big[\,\frac{x}{2}\,(t-\frac{1}{t})\Big] = \sum_{n=-\infty}^{\infty} J_n(x)t^n\ .$$

By taking the derivative $\partial/\partial\, t$ on both expressions for g, derive the recurrence relation,

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)\ .$$

[3.] Show the Laplace transform of the zeroth order Bessel function,

$$\mathcal{J}_0(a) = \int_0^\infty e^{-xa} J_0(bx)\, dx = 1\,/\,\sqrt{a^2+b^2}\ .$$

Hint: One approach is to use the series expansion of $J_0(x)$. (I used a for the transform variable to avoid confusion with the summation variable s used in class for the series index.)

[4.] Redo Problem 1 if the initial conditions instead are to set the string in motion by grabbing it in the middle ($x = L/2$) and displacing it in the shape of a triangle:

$$\begin{aligned} y(x,0) &= 2\,a\,x/L & \qquad 0 < x < L/2 \\ y(x,0) &= 2\,a\,(1-x/L) & \qquad L/2 < x < L. \end{aligned}$$

The initial velocity is zero:

$$\frac{\partial y(x,t)}{\partial t}\Big|_{t=0} = 0.$$