Physics 204A, Fall 2010, Problem Set 6

[1.] [Problem 4, UCD Qualifying Exam 2000] Obtain the Fourier expansions of the following functions:

$$\begin{aligned} f(x) &= e^{-x} \qquad 0 < x < 1 \\ f(x) &= 0 \qquad \text{otherwise} \end{aligned}$$

and

$$\begin{aligned} g(x) &= e^{-x} \qquad 0 < x < 1 \\ &= \text{extended periodically (with period 1) from } -\infty \to +\infty \end{aligned}$$

[2.] Let $F(\omega)$ be the Fourier transform of $f(x)$ and $G(\omega)$ be the Fourier transform of $g(x) = f(x+a)$. Show that

$$G(\omega) = e^{+ia\omega} F(\omega)$$

[3.] Using the sequence

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}$$

show that

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dk$$

Hint: Remember $\delta(x)$ is defined in terms of its behavior as part of an integrand.

[4.] The function $f(r)$ has a Fourier transform

$$g(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3 r = \frac{1}{(2\pi)^{3/2} k^2}$$

Determine $f(r)$. Hint: Use spherical polar coordinates in k-space.

[5.] Using partial fraction expansions, show that the inverse Laplace transform of $\frac{1}{(s+a)(s+b)}$ is $\frac{e^{-at}-e^{-bt}}{b-a}$ for $a \neq b$.

[6.] A mass m is attached to one end of an unstretched spring, spring constant k. At time $t = 0$ the free end of the spring experiences a constant acceleration a away from the mass. Using Laplace transforms, find the position x of the mass m as a function of time and determine the limiting form of $x(t)$ for small t.

[7.] Show that the Laplace transform of cosh(at)cos(at) is $s^3/(s^4 + 4a^4)$.

[8.] Solve Newton's equation for a mass m hit by an impulsive force,

$$m\frac{d^2x}{dt^2} = P\delta(t)$$

using Laplace transforms.

[9.] A random walker moves on a discrete lattice in one dimension, starting at the origin, with equal probability of hopping one step to the left and right. Show that the mean square distance $\langle n^2 \rangle$ from the origin after N steps is proportional to N.