Physics 204A, Fall 2010, Problem Set 5

[1.] Consider a damped harmonic oscillator subject to a "half-wave" driving force,

$$\begin{aligned} F(t) &= \sin\omega t & 0 < t < \pi/\omega \\ F(t) &= 0 & \pi/\omega < t < 2\pi/\omega \ . \end{aligned}$$

where $F(t)$ is periodic in t with period $T = 2\pi/\omega$. Write down the long time form for $x(t)$. Just as in the sawtooth wave problem [2b] of assignment 4, plot the $2-, 4-, 6-, 8-$ term representations of the drive force $F(t)$. Compare how well they do with the sawtooth case. Why is there a marked difference in convergence?

[2.] In class we derived a condition for a second order differential operator,

$$\mathcal{L} = p_0(t)\ d^2/dt^2 + p_1(t)\ d/dt + p_2(t)$$

to be Hermitian. As one example, $\mathcal{L} = d^2/dt^2$ is Hermitian. Is the differential operator

$$\mathcal{L} = m\ d^2/dt^2 + \gamma\ d/dt + m\omega_0^2$$

encountered in the damped oscillator problem Hermitian? What are its eigenfunctions? Are they complete?

[3.] Prove

$$\frac{1}{L}\sum_{l=1}^{L} e^{2\pi i\,(n-m)\,l\,/\,L} = \delta_{nm}$$

[4.] Show that the general solution $x(t)$ of a damped driven oscillator subject to an even force $F(t)$ (*i.e.* only cosines in the Fourier expansion) of period T is given by,

$$\begin{aligned} x(t) &= \int_0^T dt'\ \mathcal{G}(t,t')\,F(t') \\ \mathcal{G}(t,t') &= \sum_n \frac{\cos\omega_n t'\ \cos(\omega_n t - \delta_n)}{(\,m^2(\omega_0^2-\omega_n^2)^2 + \gamma^2\omega_n^2\,)^{1/2}} \\ \delta_n &= \tan^{-1}(\,\gamma\omega_n/m(\omega_0^2-\omega_n^2)\,) \end{aligned} \tag{1}$$

with $\omega_n = 2\pi n/T$. You may use all the equations we derived in class. The function $\mathcal{G}(t,t')$ is called the Green's function. This result is quite remarkable in the sense that it provides an expression for $x(t)$ which is valid for *any* $F(t)$. (This generality is the whole point of computing a Green's function.)

[5.] Similarly, prove that the general solution $\Psi(x,t)$ to a quantum problem with *any* initial wavefunction $\Psi(x,0)$ is,

$$\begin{aligned} \Psi(x,t) &= \int dx'\,\mathcal{G}(x,x',t)\,\Psi(x',0) \\ \mathcal{G}(x,x',t) &= \sum_n \phi_n^*(x')\,\phi_n(x)\,e^{-iE_nt/\hbar} \end{aligned}$$

where $H\phi_n(x) = E_n\phi_n(x)$ are the eigenfunctions and eigenvalues of the time independent Schroedinger equation.