Physics 204A, Fall 2010, Problem Set 3

[1a.] An operator $\mathcal{P}$ is said to be a "projection" if $\mathcal{P} = \mathcal{P}^2$. Prove the eigenvalues of a projection operator must be $\lambda = 0, 1$.

[1b.] A unit vector $|w\rangle$ has components w_i in a given orthonormal basis in a three dimensional space. That is, $|w\rangle = w_1|e_1\rangle + w_2|e_2\rangle + w_3|e_3\rangle$ with $\sum w_i^2 = 1$. Write the matrix (in the basis $|e_i\rangle$) representing the operator which projects any other vector $|v\rangle$ onto the plane perpendicular to $|w\rangle$. Show it obeys the result of [1a].

[2.] Redo the coupled mass-spring problem in class, but chose the masses to alternate: $M_l = M_A$ for l even and $M_l = M_B$ for l odd. *Hint:* It is fine to assume the same time dependence for $x_l(t) = v_l\, e^{i\omega t}$ as in class. However, the spatial dependence $v_l = v_0\, e^{iql}$ is no longer quite right. Can you think of a relatively small variation of this *ansatz* which recognizes that $M_A \neq M_B$, but still takes advantage of the fact that $M_1 = M_3 = M_5 = \cdots = M_B$ and $M_2 = M_4 = M_6 = \cdots = M_A$? Put another way, the system still has translation invariance, but you have to consider a "unit" consisting of a pair of masses M_A, M_B.

[3.] Show that

$$
\begin{aligned}
x_l(t) &= v_0(-1)^l \left(\frac{1-\epsilon}{1+\epsilon}\right)^l e^{i\omega t} \\
\omega^2 &= \frac{4K}{M}\frac{1}{1-\epsilon^2}
\end{aligned}
$$

is a solution of the problem of an infinite set of vibrating masses M connected by springs K and with a light defect $M' = M(1-\epsilon)$ at $l = 0$. Does this functional form make sense as $\epsilon \to 0$ and as $\epsilon \to 1$?

<u>Comment:</u> To do problems [4a], and [4b] below, you may want to keep M as part of the matrix containing the spring constant K, since it is no longer constant. Is your matrix symmetric? (If not, be careful you do not use a numerical routine for diagonalization that assumes symmetric.) You can avoid these issues by doing an alternate problem with a defect spring instead of a defect mass if you prefer.

[4a.] Solve numerically (i.e. by actually diagonalizing a matrix) for all the normal modes of a collection of $N = 128$ masses M connected by springs K. Show that you agree with the solution in class, *e.g.* verify 4-5 of the list of eigenvalues produced by the computer are correct. Also show all the participation ratios are "large", *i.e.* within a factor of 2 or so of N. *Note:* There is a subtlety here because all but two of the eigenvalues are doubly degenerate: q and $2\pi - q$ have the same ω. In such a case the eigenvectors can be arbitrary linear combinations.

[4b.] Redo [4a] (again numerically) with a single defect mass $M(1-\epsilon)$. Verify your result agrees with [3]. Plot one of the delocalized eigenvectors and also the localized eigenvector for $\epsilon = 0.1$. Show that one of your participation ratios is “small”,

[5.] In class we showed in general that stochastic matrices have eigenvalues which obey $|\lambda| \leq 1$, and that $\lambda = 1$ is one of the eigenvalues. Show this is the case for the specific matrix

$$\begin{matrix} \frac{1}{3} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \end{matrix}$$

Determine the left and right eigenvectors with eigenvalue $\lambda = 1$. Are they the same?