Physics 204A, Fall 2010, Problem Set 2

[1.] Use the definition of the Hermitian conjugate of matrices A and B to show that $(AB)^\dagger = B^\dagger A^\dagger$. (This is basically exercise A-6 of Stone and Goldbart.)

[2.] (a) Write the components C_i of $\vec{C} = \vec{A}\,\mathrm{x}\,\vec{B}$ in terms of the components of $\vec{A}$ and $\vec{B}$ using the *Levi-Civita* symbol ϵ_{ijk}.

(b) Show that $\vec{A}\cdot(\vec{A}\,\mathrm{x}\,\vec{B}) = 0$ using ϵ_{ijk}.

(c) Prove the familiar vector identity $\vec{A}\,\mathrm{x}\,(\vec{B}\,\mathrm{x}\,\vec{C}) = \vec{B}\,(\vec{A}\cdot\vec{C}) - \vec{C}\,(\vec{A}\cdot\vec{B})$ using ϵ_{ijk}.

[3.] Compute the exponential of the matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

(UCD Qualifying Exam, Fall 2010).

[4.] Show that the elements of the inverse of a matrix are related to differentiating the logarithm of its determinant:

$$\frac{\partial}{\partial a_{ij}} \ln(\mathrm{detA}) = (\mathrm{A}^{-1})_{\mathrm{ji}}$$

Although this may look like a useless, bizarre identity, it actually occurs very often in computational work. (For example, my graduate students Shan Dai and Alex Zujev needed this result to prove that the self-consistency of the mean field solution to a magnetism problem they were studying was equivalent to the requirement that the free energy be minimized.)

[5.] Consider a two dimensional space and an operator $\mathcal{A}$ which acts on two orthonormal basis vectors as follows:

$$\begin{aligned} \mathcal{A}\,|e_1\rangle &= |e_1\rangle \\ \mathcal{A}\,|e_2\rangle &= -|e_2\rangle \end{aligned}$$

Suppose we decide instead to use the vectors,

$$|f_1\rangle = (\,|e_1\rangle + |e_2\rangle\,)/\sqrt{2}$$
$$|f_2\rangle = (\,|e_1\rangle - |e_2\rangle\,)/\sqrt{2}$$

as our basis. What is the matrix for $\mathcal{A}$ in this new basis? (You will encounter these matrices, and this change in basis, when you study spin-1/2 operators in your Quantum Mechanics course.)

[6.] In problem 5 we see that very different matrices can represent the *same* operator $\mathcal{A}$, depending on the choice of basis. In Quantum Mechanics, we know that each physical observable is associated with a (Hermitian) operator. Does the fact that the matrix depends on basis imply that the physical results of a quantum problem will depend on the basis? Besides giving a general answer, illustrate your answer with the two specific matrices for $\mathcal{A}$ you found in problem 5.

[7.] Write a program to multiply the vector with components $V_n = 0.2\,n - 0.01\,n^2$ for $1 \leq n \leq 10$ by the matrix with entries $M_{nl} = 0.5 + 20/(\,n + 3\,l\,)$. What is the seventh component W_7 of $\vec{W} = M\,\vec{V}$? (There is no particular physical significance to this exercise. It merely illustrates how to set up and manipulate arrays in whatever programming language you are using. In the next homework we will look numerically at some matrices which arise in classical mechanics/phonons in solids.)