MEAN = 68

Physics 204A, Fall 2010, Midterm Exam

30 pts

[1.] Show that the sum of the squares of the elements of a matrix remains invariant under orthogonal similarity transformations.

[2.] A medium has index of refraction n which depends on position as: $n(x,y) = \sqrt{y}$. Determine the path a ray of light will travel to go from $(0,4)$ to $(6,4)$. Sketch the path. Why is it reasonable physically? What is the time taken? (You may set the speed of light $c = 1$.)

40 pts

[3.] Consider the matrix

30 pts

$$\mathcal{M} = \begin{matrix} 7 & 3 & 0 & 0 & 0 & 0 & 0 & 3 \\ 3 & 7 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 7 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 7 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 7 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 7 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 7 & 3 \\ 3 & 0 & 0 & 0 & 0 & 0 & 3 & 7 \end{matrix}$$

(a) List all the eigenvalues and eigenvectors of $\mathcal{M}$.
(b) What is $e^{-i\mathcal{M}t}|\,w\,\rangle$ where $|\,w\,\rangle$ is the column vector having all components equal to one?
(c) The matrix $\mathcal{M}$ is 'translation invariant'. That is, the matrix elements $\mathcal{M}_{jk}$ depend only on j-k (using periodic boundary conditions in interpreting j-k.) From the eigenvalues and eigenvectors of $\mathcal{M}$, write an expression for the elements of $\mathcal{M}^{-1}_{jk}$. You may leave your expression as a sum. (Do not try to evaluate it.) Show that $\mathcal{M}^{-1}$ is also 'translation invariant.
(d) Suppose the inverse matrix $\mathcal{M}^{-1}$ is raised to a high power $P = 100$. Write down the resulting matrix $[\mathcal{M}^{-1}]^P$.

P204A MT 2010 SOL'NS

$\boxed{1}$ $M' = S^T M S$

$$M'_{ij} = \sum_{\alpha\beta} S^T{}_{i\alpha} M_{\alpha\beta} S_{\beta j}$$

$$\sum_{ij} M'_{ij} M'_{ij} = \sum_{ij} \sum_{\alpha\beta} \sum_{ab} S^T{}_{i\alpha} M_{\alpha\beta} S_{\beta j} S^T{}_{ia} M_{ab} S_{bj}$$

But $S^T_{i\alpha} = S_{\alpha i}$ and can then use

$$\sum_i S^T_{i\alpha} S^T_{ia} = \sum_i S_{\alpha i} S^T_{ia} = (S S^T)_{\alpha a} = \delta_{\alpha a}$$

Likewise $S_{\beta j} = S^T_{j\beta}$ so

$$\sum_j S_{\beta j} S_{bj} = \sum_j S_{bj} S^T_{j\beta} = (SS^T)_{b\beta} = \delta_{b\beta}$$

Thus we do $\sum_{ij}$ and $\sum_{\alpha\beta}$ to get

$$\sum_{ij} M'_{ij} M'_{ij} = \sum_{ab} M_{ab} M_{ab} .$$

2.

$$dt = \frac{ds}{v} = \frac{ds}{(c/n)} = \frac{n\,ds}{c}$$

Here $n = \sqrt{y}$ so $dt = \frac{1}{c}\sqrt{y}\,ds$

Also $ds = (dx^2 + dy^2)^{1/2} = dx(1+y'^2)^{1/2}$

We want to minimize

$$\frac{1}{c}\int_{x_1}^{x_2} \sqrt{y}\,(1+y'^2)^{1/2}\,dx$$

$$f(x,y,y') = \sqrt{y}\,(1+y'^2)^{1/2}$$

No x dependence $\longrightarrow$ $f - y'\frac{\partial f}{\partial y'} = c'$

$$\sqrt{y}\,(1+y'^2)^{1/2} - y'\sqrt{y}\,\tfrac{1}{2}(1+y'^2)^{-1/2}\,2y' = c'$$

$$1+y'^2 - y'^2 = c\,(1+y'^2)^{1/2}/\sqrt{y}$$

$$1 = A\,\frac{1+y'^2}{y}$$

$$y' = \sqrt{\frac{y}{A} - 1}$$

$$\frac{dy}{\sqrt{y/A - 1}} = dx$$

$$2A\left(\frac{y}{A} - 1\right)^{1/2} = x + B$$

2 cont'd

$$\frac{Y}{A} - 1 = \left(\frac{X+B}{2A}\right)^2$$

$$Y = A + \frac{(X+B)^2}{4A}$$

To go through $(0,4)$ and $(6,4)$

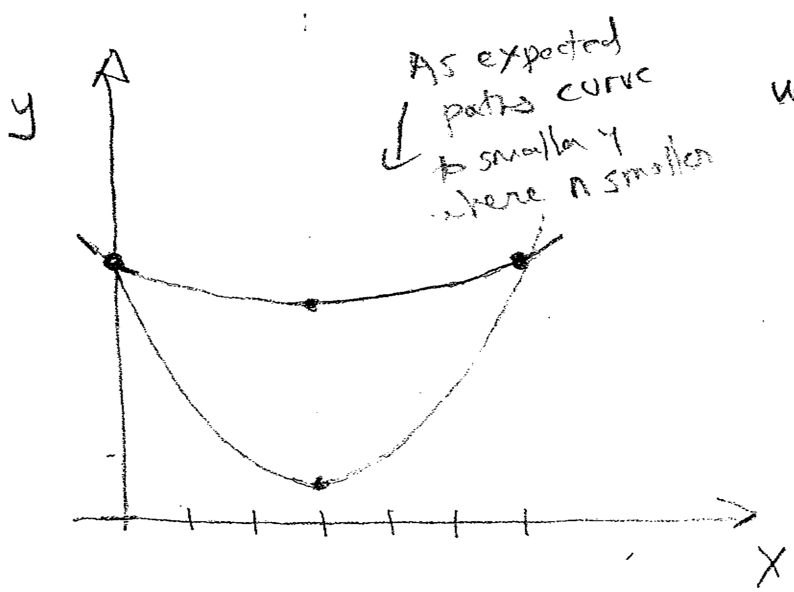
$$4 = A + B^2/4A \qquad 4 = A + \frac{(6+B)^2}{4A}$$

$$\Rightarrow \quad (6+B)^2 = B^2 \qquad 36 + 12B = 0 \qquad B = -3$$

Then $\quad 4 = A + \frac{9}{4A} \qquad 16A = 4A^2 + 9$

$$4A^2 - 16A + 9 = 0$$

$$A = \frac{16 \pm \sqrt{256 - 144}}{8} = 2 \pm \frac{\sqrt{7}}{2} = 2 \pm 1.323 = \begin{cases} 3.323 \\ 0.677 \end{cases}$$



When $x = 3$

$y = A$

which parabola gives smaller t ?

2 cont'd

$$t = \frac{1}{c}\int_0^6 \sqrt{y}\sqrt{1+y'^2}\,dx$$

$$y = A + \frac{1}{4A}(x+B)^2$$

$$y' = \frac{1}{2A}(x+B)$$

$$\sqrt{y}\sqrt{1+y'^2} = \frac{\sqrt{4A^2+(x+B)^2}}{\sqrt{4A}}\sqrt{1+\frac{1}{4A^2}(x+B)^2}$$

$$= \frac{1}{\sqrt{4A}}\frac{1}{2A}\left[4A^2+(x+B)^2\right]$$

$$t = \frac{1}{c}\frac{1}{4A\sqrt{A}}\; 4A^2x + \frac{(x+B)^3}{3}\Big|_0^6$$

recall $B=-3$

$$t = \frac{1}{c}\frac{1}{4A\sqrt{A}}\left\{(24A^2+9)-(0+9)\right\}$$

$$t = \frac{1}{c}6\sqrt{A}$$

So it appears the smaller A is preferred

$$t_{min} = \frac{1}{c}6\left[2-\frac{\sqrt{7}}{2}\right]^{1/2} = \frac{1}{c}6\sqrt{.677} = \frac{1}{c}6(.823) = \frac{4.9}{c}$$

As "sanity check" compare to straight path connecting (0,4) to (6,4)
The constant n value is $n=\sqrt{4}=2$ so the time is

$t_s = \ell/(c/n) = 1/c\;6\cdot 2 = 12/c$ ← more than twice $\frac{4.94}{c}$

3. (a) We can write the eigenequation as

$$7v_n + 3v_{n+1} + 3v_{n-1} = \lambda v_n \qquad \begin{matrix} v_0 = v_8 \\ v_1 = v_9 \end{matrix}$$

Guess soln $v_n = e^{iqn}$ $\qquad e^{iq8} = 1$

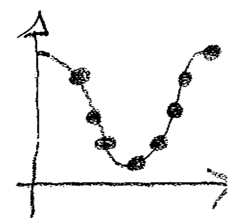
$$\boxed{\begin{matrix} 7 + 6\cos q = \lambda \\ q = \frac{\pi}{4}\{1, 2, 3, \ldots 8\} \end{matrix}}$$

Written explicitly

$$\boxed{v_n^q = \frac{1}{\sqrt{8}} e^{iqn}\,.}$$

gives the nth component of eigenvector of eigenvalue $7 + 6\cos q$

$$\lambda = \begin{cases} 7 + 3\sqrt{2} \\ 7 \\ 7 - 3\sqrt{2} \\ 1 \\ 7 - 3\sqrt{2} \\ 7 \\ 7 + 3\sqrt{2} \\ 13 \end{cases}$$



(b) later...

(c) We can write $\quad M^{-1} = \sum_q |v^q\rangle \frac{1}{\lambda_q} \langle v_q|$

$$M^{-1}_{jk} = \sum_q v^q_j \frac{1}{\lambda_q} v^{q*}_k$$

$$= \sum_q \frac{1}{\sqrt{8}} e^{iqj} \frac{1}{\lambda_q} e^{-iqk} \frac{1}{\sqrt{8}}$$

3c (cont'd)

$$(M^{-1})_{jk} = \sum_q \frac{1}{8} \frac{e^{iq(j-k)}}{\lambda_q}$$

is evidently "translation invariant".

(b) $$f(M) = \sum_q f(\lambda_q)\, |v_q\rangle\langle v_q|$$

Now the $|w\rangle$ given is just $|v_{2\pi}\rangle$ and since eigenvectors of this real symmetric matrix are orthogonal $\langle v_q|w\rangle = 0$ except for $q = 2\pi$

Thus $$e^{-iMt}|w\rangle = e^{-i\lambda_{2\pi}t}|w\rangle$$

$$= e^{-i13t}\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}$$

(d) $$M^{-1} = \sum_q \frac{1}{\lambda_q}|v_q\rangle\langle v_q|$$

Notice $\lambda_\pi = 1$ and $\lambda_q > 1$ for all other q.

Thus $\left(\frac{1}{\lambda_q}\right)^{100} \Rightarrow 0$ except $q = \pi$

$$(M^{-1})^{100} = \frac{1}{\underbrace{(\lambda_\pi)^{100}}_{1}}|v_\pi\rangle\langle v_\pi|$$

3d (cont'd)

$$(M^{-1})^{100} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 & \dots \end{pmatrix}$$

$$(M^{-1})^{100} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ & \vdots & & & & & & \end{bmatrix}$$

Is a $\underline{\underline{\text{very}}}$ good approximation. How good?

Well, how small is $\left(\frac{1}{7-3\sqrt{2}}\right)^{100} = \left(\frac{1}{2.758}\right)^{100}$

Answer: $\underline{\underline{\text{very}}}$ small!