MIDTERM EXAM, FALL 2004

Physics 204A– Mathematical Physics

[1.] (a) What are the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}?$$

(b) What matrix would you get if you diagonalized A?

(c) What is the inverse of A?

(d) Compute e^A. Hint: It is useful to take advantage of the fact that A is block diagonal.

[2.] (a) When we are expanding a function which has period 2π as a Fourier series, we are relying on the fact that the functions $\sin nx$ and $\cos mx$ are complete and orthogonal. What is a general way in which one goes about proving that a particular collection of functions form such a set?

(b) Prove orthogonality explicitly. That is, evaluate the integral,

$$\int_0^{2\pi} \sin mx \, \sin nx \; dx,$$

for integer values of n and m.

[3.] (a) Show that

$$\rho(x,t) = c\, e^{\,ikx - Dk^2 t}$$

is a solution of the diffusion equation,

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}$$

(b) What is the general solution of the diffusion equation?

(c) Suppose $\rho(x,0) = h$ for $-a < x < +a$ and $\rho(x,0) = 0$ otherwise. Write an expression for $\rho(x,t)$. It is not necessary to do the final integral over k.

(d) Suppose $h = \frac{1}{2a}$. Take the limit $a \to 0$. Now you can do the integral over k in the expression for $\rho(x,t)$ of part (c). What do you get? Have you seen the result before? When?