Mass-Spring System

\[ m \frac{d^2x}{dt^2} = -kx - \gamma \frac{dx}{dt} + f(t) \]

\[ L \frac{d^2q}{dt^2} = -\frac{q}{c} - R \frac{dq}{dt} + v(t) \]

\[ L \frac{di}{dt} = \left( -\frac{V}{R} \right) \]
Mass-Spring System

\[ m \frac{d^2 x}{dt^2} = -kx - \gamma \frac{dx}{dt} + f(t) \]

\[ m \frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + kx = f(t) \]

Suppose \( f(t) = A \cos \omega t \) (or \( A \sin \omega t \))

Also denote \( k = mw^2 \).

Guess solution \( x = B \cos(\omega t - \delta) \)

\[-m\omega^2 \cos(\omega t - \delta) + mw^2 B \cos(\omega t - \delta) \]

\[-\gamma B \omega \sin(\omega t - \delta) = A \cos \omega t \]

\[ \cos \omega t \left[ B \frac{m(w^2 - \omega^2)}{\cos \delta} + \gamma B \omega \sin \delta \right] \]

\[ + \sin \omega t \left[ B \frac{m(w^2 - \omega^2)}{\sin \delta} - \gamma B \omega \cos \delta \right] = A \cos \omega t \]

\[ \tan \delta = \frac{\omega}{m(w^2 - \omega^2)} \]

\[ B = A \left[ m(w^2 - \omega^2) \cos \delta + \gamma \omega \sin \delta \right]^{-1} \]

\[ = A \left[ \frac{m^2 (w^2 - \omega^2)^2 + \gamma^2 \omega^2}{\sqrt{m^2 (w^2 - \omega^2)^2 + \gamma^2 \omega^2}} \right]^{-1} \]

\[ = A \left[ \frac{m(w^2 - \omega^2)^2 + \gamma^2 \omega^2}{\sqrt{m^2 (w^2 - \omega^2)^2 + \gamma^2 \omega^2}} \right]^{-1/2} \]
It's important to note at this point that the differential equation we are trying to solve is linear.

So if \( x_1 \) is a solution to \( f_1 \),
\[ \hat{f} \]
\( x_2 \) is a solution to \( f_2 \),
\[ \hat{f} \]
\( x_1 + x_2 \) is a solution to \( f_1 + f_2 \).

5. We know the solution for a constant period of \( T \) of the general \( f(t) \) function is
\[ f(t) = \sum \frac{a_n \cos \frac{\omega_n t}{T}}{n} \]
\[ w_n \equiv \frac{2\pi n}{T} \]

\[ x(t) = \sum \frac{a_n \cos (w_n t - \delta_n)}{\left( m^2 (w_0^2 - w_n^2)^2 + \gamma^2 w_n^2 \right)^{1/2}} \]

\[ \delta_n = \tan^{-1} \left( \frac{\gamma w_n}{m (w_0^2 - w_n^2)} \right) \]

Is this really correct? Where does initial condition come in? Add solution of
\[ m \frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + k x = 0 \]

(Regard these as response to \( f = 0 \))
\[ x(t) = B \cos(\omega t - \delta) \]

because we will conclude \( B = 0 \) by similar algebra as before.

(Recall \( B \) is amplitude of driving force)

What can we do? \( x(t) = B e^{-\alpha t} \cos(\omega t - \delta) \)

Do we expect \( \omega = \omega_0 \)? \( \omega = \omega_1 \) \( \omega < \omega_0 \) \( \omega > \omega_0 \)?

A little bit of algebra... (MS-3A)

\[ x = \frac{y}{2m} \]

\[ \omega^2 = \omega_0^2 - \frac{y^2}{4m^2} \]

\[ x(t) = B e^{-\frac{y}{2m} t} \cos \left( \sqrt{\omega_0^2 - \frac{y^2}{4m^2}} \ t - \delta \right) \]

\( B, \delta \rightarrow x(0), y(0) \)

What happens as \( t \rightarrow 00 \)?
\[ x = -\alpha e^{-\alpha t} \cos(\omega t - \delta) - \omega e^{-\alpha t} \sin(\omega t - \delta) \]

\[ x = \alpha^2 e^{-\alpha t} \cos(\omega t - \delta) + \omega \omega e^{-\alpha t} \sin(\omega t - \delta) \]

\[ -\omega^2 e^{-\alpha t} \cos(\omega t - \delta) \]

\[ e^{-\alpha t} \text{ cancels everywhere} \]

\[ \cos(\omega t - \delta) \left[ m\alpha^2 - m\omega^2 - 3\alpha + m\omega_0^2 \right] + \sin(\omega t - \delta) \left[ 2m\omega \alpha - \delta \omega \right] = 0 \]

\[ \alpha = \frac{\omega}{2m} \]

\[ \omega^2 = \omega_0^2 + \alpha^2 - \frac{\gamma}{m} \alpha \]

\[ = \omega_0^2 - \frac{\gamma}{2m} \]

\[ \omega < \omega_0 \]
Summary:

Original problem \[ \rightarrow \text{HARD} \rightarrow \text{solution of original problem} \]

Problem in transform space \[ \rightarrow \text{EASY} \rightarrow \text{solution in transform space} \]

Fourier, Laplace, ...
Resonance +
Most of response function:

\[
\left( m^2 (w_0^2 - w_n^2) + y^2 w_n^2 \right)^{1/2}
\]

\[
w_n = \frac{2\pi n}{T}
\]

\(\text{Note: unique and thus has maximum when } w_n = w_0\)

\[
d/w_n \Rightarrow \left[ m^2 (w_0^2 - w_n^2)^2 + y^2 w_n^2 \right]^{3/2} \left( -1/2 \right)
\]

\[
\left[ 2 m^2 (w_0^2 - w_n^2) (-2w_n) + 2 y^2 w_n \right]
\]

\[
f'(w_n) = \frac{2 m^2 w_n (w_0^2 - w_n^2) - y^2 w_n}{\left[ m^2 (w_0^2 - w_n^2)^2 + y^2 w_n^2 \right]^{3/2}} = 0\]

\[
\text{Maxima p126-7:}
\]

\[
121 = \left[ R^2 + (4t - 4c)^2 \right]^{1/2}
\]

\[
x^2 + y^2 = \text{max}w^2x = 0
\]

\[
L^2 + R^2 + 2c Q = 0
\]

\[
121 = \left( w_m - w_m w_0^2 \right)^2
\]

\[
= \sqrt{w_0^2 + m^2 (w_0^2 - w_n^2)^2}
\]

\[
= \frac{1}{w_n^2} \left[ y^2 w_n^2 + m^2 (w_0^2 - w_n^2)^2 \right]
\]

Resonance condition is not \( w_n = w_0 \)

But \( w_0 \) because natural frequency is not \( w_0 \)

w_0 is incorrect!
Question in class:

If Amplitude remains small, where does Energy go?

We gave correct qualitative answer: When \( f(t) \) and \( x(t) \) not in phase, Mass often does work on object except \( f \), Net work done is small.

More quantitative:

\[
\text{Power} = \frac{1}{T} \int_{0}^{T} f(t) \overline{x(t)} \, dt
\]

\[
= \frac{1}{T} A B \omega_n \int_{0}^{T} \cos(t) \sin(\omega_n t) - \cos(t) \sin(\omega_n t) \, dt
\]

\[
= \frac{1}{2} A B \omega_n \frac{T}{2} \sin\delta
\]

\[
= \frac{1}{2} \frac{A^2}{m} \sin^2 \delta
\]

But \( B = \frac{A}{\sqrt{m^2 (\omega_n^2 - \omega^2) + \gamma^2 \omega_n^2}} \)

\[
\sin \delta = \frac{\omega_n}{\sqrt{\omega_n^2 + \gamma^2 (\omega_n^2 - \omega^2)^2}}
\]

\[
\cos \delta = \frac{\gamma \omega_n}{\sqrt{\omega_n^2 + \gamma^2 (\omega_n^2 - \omega^2)^2}}
\]

\[
\sin \delta = \frac{\omega_n}{\sqrt{\omega_n^2 + \gamma^2 (\omega_n^2 - \omega^2)^2}}
\]

1. \( \omega_n = \gamma \):

\[\gamma = \alpha \quad \text{no work done,} \]

\[
\text{Power} = \frac{1}{2\alpha} A^2 \sin^2 \delta
\]
\[ w = w_0 \quad \sin \delta = 1 \]

Resonance of power and amplitude of which are not the same, DO and not same as frequency.

- Resonance of power: \[ w = w_0 \]
- Resonance of amplitude: \[ w^2 = (w_0^2 - \frac{\delta^2}{2m^2})^{1/2} \]
- Natural frequency: \[ w^2 = (w_0^2 - \frac{\delta^2}{4m^2})^{1/2} \]

\[ w = w_0 \quad \frac{1}{\sqrt{1 + \frac{m^2}{\gamma^2}(\frac{\delta}{w_0})^2}} \quad m/\gamma = 100 \]