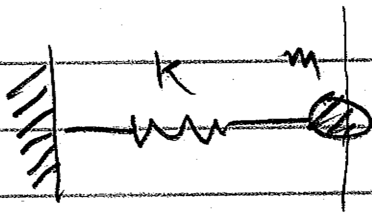
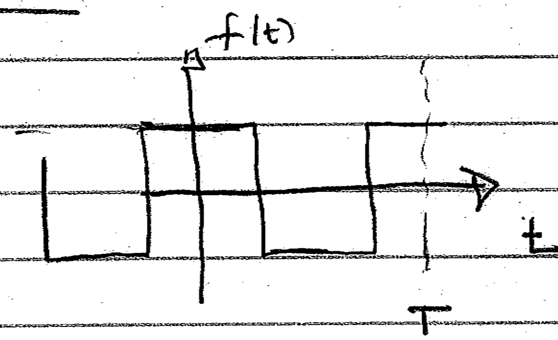
Ms-0

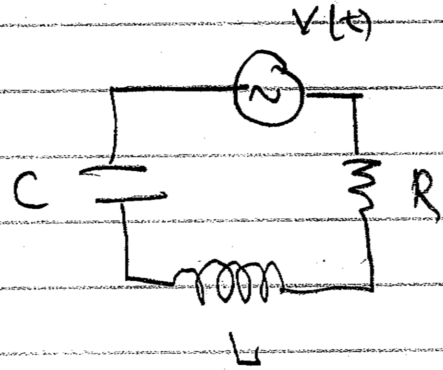
Mass - Spring system



friction



or any periodic function of period T



completely equivalent

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt} + f(t)$$

$$\underbrace{L\frac{d^2Q}{dt^2}}_{L\frac{dI}{dt}} = -\frac{Q}{c} - \underbrace{R\frac{dQ}{dt}}_{IR} + V(t)$$

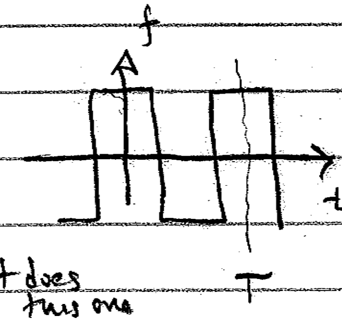
MS-1

Mass-Spring System

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt} + f(t)$$

friction

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = f(t)$$



text does this one

Suppose $f(t) = A\cos\omega t$ (or $A\sin\omega t$)

Also denote $k = m\omega_0^2$. Guess sol'n $x = B\cos(\omega t - \delta)$

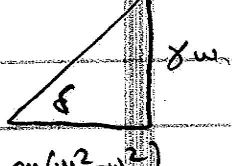
$$-mB\omega^2\cos(\omega t-\delta) + m\omega_0^2 B\cos(\omega t - \delta)$$

$$-\gamma B\omega\sin(\omega t - \delta) = A\cos\omega t$$

work through w/ $m\, e^{i\omega t}$

$$\cos\omega t\left[B\,m(\omega_0^2-\omega^2)\cos\delta + \gamma B\omega\sin\delta\right]$$
$$+ \sin\omega t\left[B\,m(\omega_0^2-\omega^2)\sin\delta - \gamma B\omega\cos\delta\right] = A\cos\omega t$$

$$\tan\delta = \frac{\gamma\omega}{m(\omega_0^2-\omega^2)}$$ makes $\sin\omega t$ term vanish on rhs.



$$B = A\left[m(\omega_0^2-\omega^2)\cos\delta + \gamma\omega\sin\delta\right]^{-1}$$

$$= A\left[\frac{m^2(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}}\right]^{-1}$$

$$= A\left[m^2(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2\right]^{+1/2}$$

MS-2

Impt to note at this point that the diff eqn we are trying to solve is <u>linear</u>

So if x_1 is soln to f_1
& x_2 is soln to f_2

$x_1 + x_2$ is soln to $f_1 + f_2$

8. We know soln for $A \cos \omega t$

obvious strategy for general $f(t)$ of period T then is

$$f(t) = \sum_n a_n \cos \frac{2\pi n t}{T} \qquad \omega_n \equiv \frac{2\pi n}{T}$$

$$x(t) = \sum_n \frac{a_n \cos(\omega_n t - \delta_n)}{\left(m^2(\omega_0^2 - \omega_n^2)^2 + \gamma^2 \omega_n^2\right)^{1/2}}$$

$$\delta_n = \tan^{-1}\left(\gamma\omega_n / m(\omega_0^2 - \omega_n^2) \right)$$

Is this really correct? Where do initial conditions come in? Add soln of

$$m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0$$

(Regard this as response to $f = 0$)

MS-3

Guess soln $x(t) = B \cos(\omega t - \delta)$

because will conclude $B = 0$ by similar algebra to before.

(Recall $B \sim$ amplitude of driving force)

What can we do? $x(t) = B e^{-\alpha t} \cos(\omega t - \delta)$

Do we expect $\omega = \omega_0$? $\omega = \omega_0$ ([illegible])

$\omega < \omega_0$? $\omega > \omega_0$?

A little bit of algebra ... (MS-3A)

$\alpha = \gamma/2m$

$\omega^2 = \omega_0^2 - \gamma^2/4m^2$

$$x(t) = B e^{-(\gamma/2m)t} \cos\left(\sqrt{\omega_0^2 - \frac{\gamma^2}{4m^2}}\, t - \delta\right)$$

$B, \delta \longleftrightarrow x(0); v(0)$

What happens as $t \to \infty$?

MS-3A

B cancels everywhere

$$\dot{x} = -\alpha e^{-\alpha t}\cos(\omega t-\delta) - \omega e^{-\alpha t}\sin(\omega t-\delta)$$

$$\ddot{x} = \alpha^2 e^{-\alpha t}\cos(\omega t-\delta) + 2\omega\alpha e^{-\alpha t}\sin(\omega t-\delta)$$

$$- \omega^2 e^{-\alpha t}\cos(\omega t-\delta)$$

$e^{-\alpha t}$ cancels everywhere

$$\cos(\omega t-\delta)\left[m\alpha^2 - m\omega^2 - \gamma\alpha + m\omega_0^2\right] = 0$$

$$+ \sin(\omega t-\delta)\left[2m\omega\alpha - \gamma\omega\right]$$

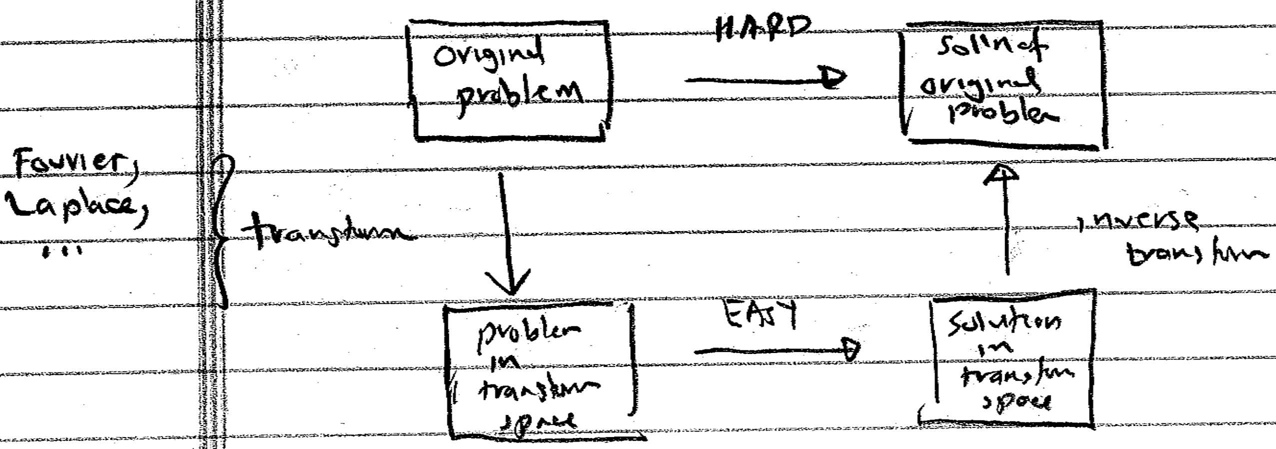
$$\alpha = \gamma/2m$$

$$\omega^2 = \omega_0^2 + \alpha^2 - \frac{\gamma}{m}\alpha$$
$$= \omega_0^2 + \gamma^2/4m^2 - \gamma^2/2m^2 = \omega_0^2 - \gamma^2/4m^2$$

$$\omega < \omega_0$$

MS-4

Summary:


Original problem
HARD
Sol'n of original problem
Fourier, Laplace, ...
transform
inverse transform
problem in transform space
EASY
Solution in transform space

MS-2'

Resonance +
Max of response function:

$$\left(m^2(\omega_0^2-\omega_n^2)^2+\gamma^2\omega_n^2\right)^{-1/2}$$

$$\omega_n = 2\pi n / T$$

Check Marion p10 2-3

$m\ddot{x}+b\dot{x}+kx=0$

$\ddot{x}+2\beta\dot{x}+\omega_0^2 x=0$

$\beta = \gamma/2m$ ✓

$\omega_1 = \omega_0^2-\beta^2$

$= \omega_0^2-\gamma^2/4m^2$ ✓✓

my p MS-3A

Want to argue that this has maximum when $\omega_n = \omega_0$

$$d/d\omega_n \Rightarrow \left[m^2(\omega_0^2-\omega_n^2)^2+\gamma^2\omega_n^2\right]^{-3/2}(-1/2)$$

$$\left[2m^2(\omega_0^2-\omega_n^2)(-2\omega_n)+2\gamma^2\omega_n\right]$$

$$f'(\omega_n) = \frac{2m^2\omega_n(\omega_0^2-\omega_n^2)-\gamma^2\omega_n}{\left[m^2(\omega_0^2-\omega_n^2)^2+\gamma^2\omega_n^2\right]^{3/2}} = 0$$

Check Marion p126-7

$|Z| = \left[R^2+(\omega L-1/\omega C)^2\right]^{1/2}$

$m\ddot{x}+\gamma\dot{x}+m\omega_0^2 x=0$

$L\ddot{Q}+R\dot{Q}+1/C\, Q=0$

$\rightarrow |Z|^2 = \gamma^2+(\omega m-1/\omega\, m\omega_0^2)^2$

$= \gamma^2+\frac{m^2}{\omega^2}(\omega^2-\omega_0^2)^2$

$= \frac{1}{\omega^2}\left[\gamma^2\omega^2+m^2(\omega^2-\omega_0^2)^2\right]$

$$\omega_n\left[2m^2(\omega_0^2-\omega_n^2)-\gamma^2\right]=0$$

$$\omega_n = 0 \quad \text{or}$$

$$\omega_n^2 = \frac{-\gamma^2}{2m^2}+\omega_0^2$$

Resonance condition is <u>not</u> $\omega_n = \omega_0$

<u>But</u> that's because natural frequency is not ω_0 when friction is present!

PF-1

Power factor

Question in class:

If Amplitude remains small, where does Energy go?

We gave correct qualitative answer: when $f(t)$ and $x(t)$ not in phase, Mass often does work on object exerting f, Net work done is small

More quantitative

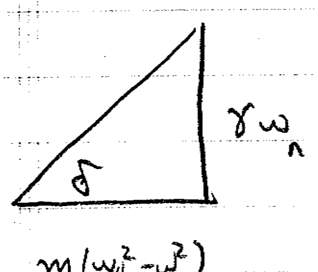
$$\text{Power} = \frac{1}{T}\int_0^T \overbrace{A\cos\omega_n t}^{f(t)} \;\overbrace{\left(-B\omega_n \sin(\omega_n t - \delta)\right)}^{dx} dt$$

$$= \frac{-1}{T} AB\omega_n \int_0^T \cos\omega t \left(\sin\omega_n t \cos\delta - \cos\omega_n t \sin\delta\right) dt$$

$$= +\frac{1}{T} AB\omega_n \frac{T}{2} \sin\delta \qquad \text{but } B = \frac{A}{\left[m^2(\omega_0^2-\omega_n^2)^2+\gamma^2\omega_n^2\right]^{1/2}}$$

$$= \frac{1}{2}\frac{A^2}{\gamma} \sin^2\delta$$

"power factor"



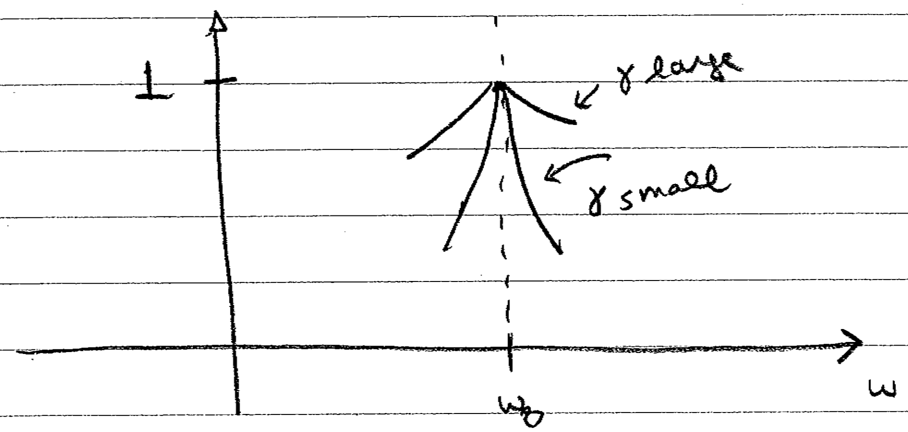
$$\sin\delta = \frac{\gamma\omega_n}{\left[\gamma^2\omega_n^2 + m^2(\omega_0^2-\omega_n^2)^2\right]^{1/2}}$$

limits: $\gamma = 0$ no work done!

$$\text{Power} = \frac{1}{2\gamma} A^2 \sin^2\delta$$

(PF - 2)

$\omega = \omega_0 \quad \sin\delta = 1$



Resonance of power and amplitude of motion are not the same DO and not same as frequency

$\omega = \omega_0$ ← resonance of power
$\omega = (\omega_0^2 - \gamma^2/2m^2)^{1/2}$ ← resonance of amplitude
$\omega = (\omega_0^2 - \gamma^2/4m^2)^{1/2}$ ← natural frequency

$$\gamma\omega \Big/ \left[\gamma^2\omega^2 + m^2(\omega_0^2 - \omega^2)^2\right]^{1/2} = \frac{1}{\left[1 + \frac{m^2\omega^2}{\gamma^2}\left(1 - \frac{\omega_0^2}{\omega^2}\right)^2\right]^{1/2}}$$

$\omega = \omega_0 \qquad 1$

$\omega = 1.1\omega_0 \qquad 1\Big/\sqrt{1 + \frac{m^2}{\gamma^2}(.04)} \qquad m/\gamma = 100$