0-A

Laplace Transforms

Conditions for existence:

$x \to \infty$ $f(x)$ can diverge for large x as long as there is some s_0 such that $e^{-s_0 x} f(x)$ is finite as $x \to \infty$. Then the Laplace transform will exist for $s > s_0$.

$x \to 0$ $f(x)$ cannot diverge too fast at $x \to 0$. Eg Laplace transform of x^n doesn't exist if $n \leq -1$.

More
- e^{x^2} has no Laplace transform
- x^n is okay $e^{-sx} x^n \to 0$ as $x \to \infty$ $\forall$ $s > s_0 = 0$
- Laplace transform of $e^{ax} \to 1/s-a$ well defined for $s > a$.

Alternately

$$F(s) \equiv \lim_{x_{max} \to \infty} \int_0^{x_{max}} e^{-sx} f(x) dx$$

phrase in terms of QM!

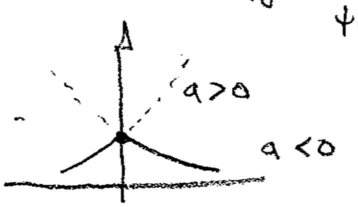
$F(s)$ is defined for a particular s if limit exists.

$\to$ Avoided this issue for F.T. because periodic on $[0, L]$

But same problem would occur after $L \to \infty$ (Fourier integral)

eg. $f(x) = e^{a|x|}$

$\psi(x)$



$$c(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} e^{a|x|}$$

$\boxed{6\text{-}13}$

$$c(k) = \frac{1}{\sqrt{2\pi}}\left[\int_0^\infty e^{-ikx} e^{ax}\,dx + \int_{-\infty}^0 e^{-ikx} e^{-ax}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{ \frac{1}{a-ik} e^{(a-ik)x}\Big|_0^\infty + \frac{1}{-a-ik} e^{-(a+ik)x}\Big|_{-\infty}^0 \right\}$$

only if $a<0$!

$$= \frac{1}{\sqrt{2\pi}}\left\{ \frac{-1}{a-ik} + \frac{1}{-a-ik} \right\}$$

$$= \frac{-1}{\sqrt{2\pi}}\left\{ \frac{1}{a-ik} + \frac{1}{a+ik} \right\} = \frac{-1}{\sqrt{2\pi}}\left\{ \frac{a+ik+a-ik}{a^2+k^2} \right\}$$

$$c(k) = -\frac{2}{\sqrt{2\pi}} \frac{a}{a^2+k^2}$$ ← recall $a<0$! so $c(k)>0$

Difference though appears to be that onus is entirely on original $f(x)$. If $f(x)$ obeys certain conditions, eg $a<0$, $c(k)$ is defined $\forall k$.

For Laplace case $F(s)$ exists depending not only on $f(x)$ but also on value of s, eg $F(s)$ exists for $s>a$ if $f(x) = e^{ax}$.

It's clear why this is so mathematically:
e^{-sx} can control some of bad behavior of $f(x) = e^{ax}$ if s is large enough

Still, physical interpretation awaits inversion formula $f(x) = \ldots F(s) \ldots$ in turn awaits integration in complex plane

0-c

Numerical
Laplace transform of 1

$$F(s) = \int_0^\infty dx\, e^{-sx} = 1/s$$

Means can construct 1 via putting together

$s = 0.1, 0.2, 0.3, \ldots 1.0$
$1/s = 10, 5, 3.3, \ldots 1$

$$f(x) = 1 = (10e^{-.1x} + 5e^{-.2x} + \quad e^{-x}) / [10+5+3.3+\ldots 1]$$

general $s_i = ds \cdot i$ $\quad i = 1, 2, 3, \ldots L/ds$

ds	L	f(1)	f(2)
.1	1	.734	.569
.2	2	.569	.378
.1	2	.641	.474
.05	2	.695	.549
.02	2	.747	.453
.01	2	.776	.667
.004	2	.806	.711
.0002	2	.865	.799

not too good!