L-1

Laplace Transforms

FOURIER $\quad c(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-ikx} f(x)\,dx$

LAPLACE $\quad F(s) = \int_{0}^{\infty} dx\, e^{-sx} f(s)\,dx$

You may have thought already

$\frac{d^2}{dx^2}$ Hermitian operator eigenvectors $e^{\pm ikx}$ $\begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix}$

why not just one derivative?

$\frac{d}{dx} e^{-sx} = -s\, e^{-sx}$

$\langle f | \tfrac{d}{dx} | g \rangle$
$= \int f(x) \frac{d}{dx} g(x)\, dx$
$= -\int \tfrac{d}{dx} f(x)\, g(x) + \text{bdy}$
$= -\langle g | \tfrac{d}{dx} | f \rangle$ not Hermitian

$f(x) = 1$

$$F(s) = \int_0^{\infty} dx\, e^{-sx} = \left.\frac{e^{-sx}}{-s}\right|_0^{\infty} = 1/s$$

$f(x) = e^{xa}$

$$F(s) = \int_0^{\infty} dx\, e^{-sx} e^{xa} = \left.\frac{1}{a-s} e^{-x(s-a)}\right|_0^{\infty} = \frac{1}{s-a}$$

Laplace transforms are linear

$F(x) = \cosh xa = \frac{e^{xa} + e^{-xa}}{2}$ $\quad F(s) = \left(\frac{1}{s-a} + \frac{1}{s+a}\right)\frac{1}{2} = \frac{s}{s^2-a^2}$

$F(x) = \sinh xa = \frac{e^{xa} - e^{-xa}}{2}$ $\quad F(s) = \left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2-a^2}$

$\boxed{L-1'}$

$$\left.\begin{array}{l} f(x) = \cos xa \\ \dfrac{e^{ixa} + e^{-ixa}}{2} \end{array}\right\} \quad F(s) = \left(\frac{1}{s-ia} + \frac{1}{s+ia}\right)\frac{1}{2} = \frac{s}{s^2+a^2}$$

$$f(x) = \sin xa \qquad F(s) = \frac{a}{s^2+a^2}$$

L-2

$$f(x) = x^n$$

$$F(s) = \int_0^\infty \underbrace{dx\, e^{-sx}}_{du} \underbrace{x^n}_{v}$$

$$= \frac{e^{-sx}}{-s} x^n \Big|_0^\infty + \int_0^\infty dx \frac{e^{-sx}}{s} n x^{n-1}$$

$$= \frac{n}{s} \int_0^\infty e^{-xs} x^{n-1} dx$$

$$= \frac{n(n-1)}{s^2} \int_0^\infty e^{-xs} x^{n-2} dx$$

$$= \ldots = n!/s^{(n+1)}$$

Inversion of Laplace transforms ...

Two approaches
① Recognize as ...
② More formal: will have to wait!

Laplace transform of derivative

$$f(x) \rightarrow F(s) = \int_0^\infty dx\, e^{-sx} f(x)$$

$$f'(x) \rightarrow \int_0^\infty dx\, e^{-sx} f'(x)$$

$$= f(x) e^{-sx} \Big|_0^\infty - \int_0^\infty dx \left(-s e^{-sx}\right) f(x)$$

$$= -f(0) + s F(s)$$

check $\quad f(x) = x^n \rightarrow F(s) = n!/s^n$

$$f'(x) = n x^{n-1} \rightarrow -(0) + s\, n!/s^n = n!/s^{n-1}$$

from then

$$= n\, (n-1)!/s^{n-1}$$ directly ✓✓

$$f''(x) \rightarrow -f'(0) + s\left(-f(0) + sF(s)\right)$$
$$= -f'(0) - sf(0) + s^2 F(s)$$

L-21

$$f(x) \to F(s)$$

pick up factor of s

$$f'(x) \to -f(0) + sF(s)$$

Perhaps no surprise that

$$\int_0^x f(x')dx' \to \frac{1}{s} F(s)$$

proof

$$g(x) = \int_0^x f(x')dx'$$

$$g'(x) = f(x)$$

since $g(0)=0$

$$\therefore \quad g'(x) \to F(s) = -g(0) + F(s)$$

compare

$$g'(x) \qquad = -g(0) + s\,G(s)$$

$$G(s) = 1/s\, F(s)$$

L-2''

SIMPLE EXERCISE

$$df/dx = a f$$

$$-f(0) + sF(s) = a F(s)$$

$$F(s) = f(0)/s-a$$

$$f(x) = f(0)\, e^{+ax}$$

Similarly for $$d^2f/dx^2 = a^2 f$$

$$-f'(0) - s f(0) + s^2 F = a^2 F$$

$$F = \frac{f'(0)}{s^2-a^2} + \frac{s\, f(0)}{s^2-a^2}$$

$$f = \frac{f'(0)}{a} \sinh xa + f(0) \cosh xa$$

L-3

What is Laplace transform of $e^{-bx} f(x)$

If Laplace transform of $f(x)$ is $F(s)$?

$$\int_0^\infty e^{-sx} e^{-bx} f(x)\,dx = \int_0^\infty e^{-(s+b)x} f(x)\,dx$$

$$= F(s+b) \quad \text{"shift theorem"}$$

Application:

Laplace transform of $e^{-bx} \sin ax$ is $\dfrac{a}{(s+b)^2 + a^2}$

$e^{-bx} \cos ax$ is $\dfrac{s+b}{(s+b)^2 + a^2}$

L-4

Damped oscillator (again)

$f \to x \qquad f$

$x \to t$

$f(x) \to x(t)$

leave as $M\frac{d^2f}{dx^2} + \gamma\frac{df}{dx} + kf = 0$

$$M\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0$$

solve for time dependent force shortly did it already by Fourier...

$$m\left[-x'(0) - s\,x(0) + s^2\,\bar{X}(s)\right] + \gamma\left[-x(0) + s\,\bar{X}(s)\right] + k\,\bar{X}(s) = 0$$

$$\bar{X}(s) = \frac{(ms+\gamma)x_0 + mv_0}{ms^2 + \gamma s + k}$$

$k = m\omega_0^2$

$\gamma = 2mb$

$$\bar{X}(s) = \frac{(s+2b)x_0 + v_0}{s^2 + 2bs + \omega_0^2}$$

$$s_\pm = \frac{1}{2}\left[-2b \pm \sqrt{4b^2 - 4\omega_0^2}\right] \qquad \text{Assume } \omega_0 > b$$

$$= -b \pm i\sqrt{\omega_0^2 - b^2}$$

real

L-5

Complete square in denominator

$$s^2 + 2bs + \omega_0^2 = (s+b)^2 + \underbrace{\omega_0^2 - b^2}_{\equiv\, \omega_1^2}$$

$$\bar{X}(s) = \frac{(s+b)x_0}{(s+b)^2 + \omega_1^2} + \frac{bx_0 + v_0}{(s+b)^2 + \omega_1^2}$$

$$x(t) = x_0\, e^{-bt} \cos\omega_1 t + (bx_0 + v_0)\frac{1}{\omega_1} e^{-bt} \sin\omega_1 t$$

check $\quad x(0) = x_0$

$$v(t) = x_0\, e^{-bt}(-\omega_1 \sin\omega_1 t) - bx_0 e^{-bt}\cos\omega_1 t$$

$$+ (bx_0 + v_0)\left(\frac{-b}{\omega_1}\right) e^{-bt} \sin\omega_1 t$$

$$+ (bx_0 + v_0)\, e^{-bt} \cos\omega_1 t$$

$$v(0) = -bx_0 + bx_0 + v_0 \quad \checkmark\checkmark$$

Note: Laplace transform soln of diff eqn builds initial conditions in automatically.

$\boxed{L-6}$

We know

$$f(x) \to F(s)$$
$$f'(x) \to -f(0) + sF(s)$$
$$f''(x) \to -f'(0) - sf(0) + s^2 F(s)$$
$$\int_0^x f(x')dx' \to F(s)/s$$

What about $\quad ? \leftarrow dF(s)/ds$

Consider $\quad F(s) = \int_0^\infty e^{-sx} f(x)dx$

$$\frac{dF(s)}{ds} = \int_0^\infty -xe^{-sx} f(x)dx$$

$$= \int_0^\infty e^{-sx}(-xf(x))\,dx$$

$\therefore \quad -xf(x) \leftarrow dF(s)/ds$

Similarly $\quad x^2 f(x) \leftarrow d^2F/ds^2$

Laplace transform of sin x

Laplace transform of $x\sin x$ is $-d/ds\left[\;1/s^2+1\;\right]$

$$= 2s/(s^2+1)^2$$

$\boxed{L-7}$

$? \leftarrow 1/(s^2+1)^2$

$$1/(s^2+1)^2 = \frac{-1}{2s} \, d/ds \, 1/(s^2+1)$$

$$\Rightarrow -1/2 \int_0^x dx' \, (-x' \sin x')$$

$1/s$ means must integrate

d/ds means $-x$ times

$$= 1/2 \int_0^x \underbrace{x'}_{v} \underbrace{\sin x' \, dx'}_{du}$$

$$= 1/2 \left[-x' \cos x' \Big|_0^x + \int_0^x \cos x' \, dx' \right]$$

$$= 1/2 \left[x \cos x - \sin x' \Big|_0^x \right]$$

$$= 1/2 (x \cos x - \sin x)$$

$\boxed{L\text{-}8}$

What if $f(x)$ is periodic on $(0, L)$?

$$\mathcal{L}[f(x)] \equiv \int_0^\infty e^{-sx} f(x)\,dx$$

$$= \int_0^L e^{-sx} f(x)\,dx + \int_L^{2L} e^{-sx} f(x)\,dx + \int_{2L}^{3L} e^{-sx} f(x)\,dx + \ldots$$

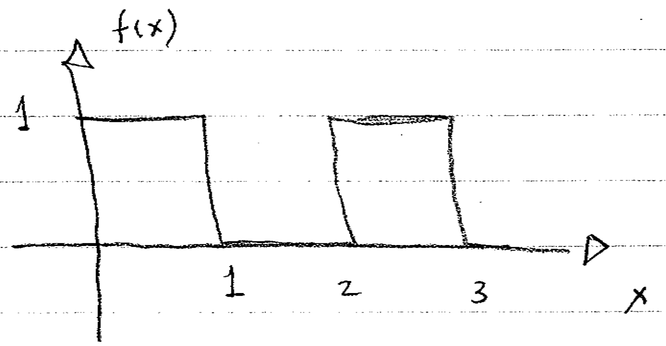
$y = x - L$

$x = y + L$

$$= \int_0^L e^{-sx} f(x)\,dx + e^{-sL}\int_0^L e^{-sy} f(y)\,dy + \ldots$$

$$= \left[1 + e^{-sL} + e^{-2sL} + \ldots\right] \int_0^L e^{-sx} f(x)\,dx$$

$$= \left[\frac{1}{1 - e^{-sL}}\right] \int_0^L e^{-sx} f(x)\,dx$$



$L = 2$

$$\int_0^2 e^{-sx} f(x)\,dx = \int_0^1 e^{-sx}\,dx = \left.\frac{e^{-sx}}{-s}\right|_0^1 = \frac{1 - e^{-s}}{s}$$

$$\rightarrow \frac{1}{1 - e^{-2s}} \frac{1 - e^{-s}}{s} = \frac{1}{s(1 + e^{-s})}$$

$\boxed{L-9}$

$$d^2 f/dx^2 + f = x$$

Method #1 Laplace transform

$$-f'(0) - sf(0) + s^2 F + F = 1/s^2$$

STRATEGY

DE $\rightarrow$ ALGEBRA

BUT THEN MUST DO INVERSE TRANSFORM

$$(s^2+1)F = 1/s^2 + sf(0) + f'(0)$$

$$F = \frac{f'(0)}{s^2+1} + f(0)\frac{s}{s^2+1} + \underbrace{\frac{1}{s^2}\frac{1}{(s^2+1)}}$$

$$\Rightarrow f'(0)\sin x + f(0)\cos x$$

$$1/s^2 - 1/s^2+1$$

$$\Rightarrow x - \sin x$$

$$f(x) = x + [f'(0) - 1]\sin x + f(0)\cos x$$

Method #2 solve homogeneous eqn generally

$$f(x) = A\sin x + B\cos x$$

Find any soln of inhomogeneous eqn

$f(x) = x$ obviously

$$f(x) = x + A\sin x + B\cos x \qquad f'(x) = 1 + A\cos x - B\sin x$$

$$f(0) = B \qquad f'(0) = 1 + A$$

Challenge

$$\frac{d^2f}{dx^2} + 4\frac{df}{dx} + 13f = 2x + 3e^{-2x}\cos 3x$$

$$f(0) = 0 \qquad f'(0) = -1$$

Green's functions and Laplace transforms

We encountered Green's functions when we solved diffusion eqn. Now they'll appear again.

(Finney + Ostberg 237-260)