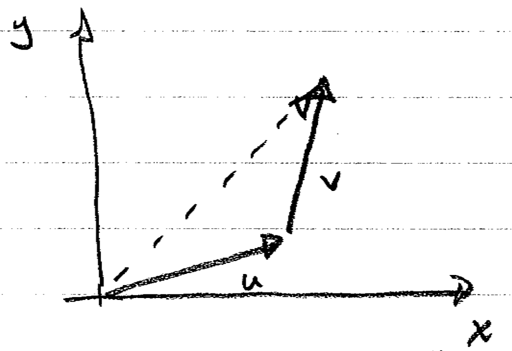
Vectors

First let's discuss what a vector is.

When we first encounter them, vectors are arrows in $d = 2, 3$



vectors in this context have length given by
$|u|^2 = u_x^2 + u_y^2$

We define a rule "addition" to combine them

$u + v$

And also we define a dot product

$$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y = |u||v| \cos\theta$$

Note
$|u|^2 = \vec{u} \cdot \vec{u}$

Neither $|u|^2$ nor $\vec{u} \cdot \vec{v}$ depend on the choice of axes.
(We talked about this a bit)

Addition obeyed certain rules

$u + v = v + u$

$\exists\, 0 \qquad 0 + u = u$

$\exists\, -u \qquad u + -u = 0$

$u + (v + w) = (u + v) + w$

as did the dot product $\quad \vec{u} \cdot \vec{u} \geq 0 \quad \vec{u} \cdot \vec{u} = 0$ iff $\vec{u} = 0$

The key observation is that functions obey all these rules too. In fact, we generalize the notion of a vector (and vector space) to be a group of objects and an "addition" rule and a dot product which has all the properties of "familiar" vectors.

Does anyone know what addition rule is? Almost a silly question, the answer is so natural

$$(f+g)(x) = f(x)+g(x)$$

$$0(x) = 0 \qquad \text{etc.}$$

What about dot product?

$$\langle f|g\rangle = \int_a^b f^*(x)g(x)\,dx$$

(a, b might be $(-\infty,\infty)$)

$\rightarrow \int_a^b f(x)g(x)\,dx$ is considering only real functions

$$|f|^2 = \langle f|f\rangle$$

$$= \int_a^b dx\, \underbrace{f^*(x)f(x)}_{\text{nonnegative}} \geq 0$$

Y3

Schwarz inequality

$$|\vec{u}\cdot\vec{v}|^2 = |u|^2|v|^2\cos^2\theta \le |u|^2|v|^2$$

consider $(a,b) = (0,2)$ and $f = 3 + \cancel{x}x$

$g = 2x^2$

$$\left|\int dx\, f(x)g(x)\right|^2 \le \int dx\, |f(x)|^2 \int dx\, |g(x)|^2$$

$$\int_0^2 dx\, (3+x)(2x^2) \qquad \int dx\, (3+x)^2 \qquad \int (2x^2)^2 dx$$

$$\int_0^2 dx\, (6x^2 + 2x^3) \qquad \int dx\; 9+6x+x^2 \qquad \int dx\; 4x^4$$

$$2x^3 + x^4/2 \Big|_0^2 \qquad 9x+3x^2+x^3/3 \qquad 4x^5/5$$

$$(24)^2 \le (32\,^2/_3) \quad (^{128}/_5)$$

$$24\cdot 24 \le 32.666 \quad 25.60$$

$$576 \le 836.27$$