Fourier Series

What is it? Expand any f(x) which has period 2π

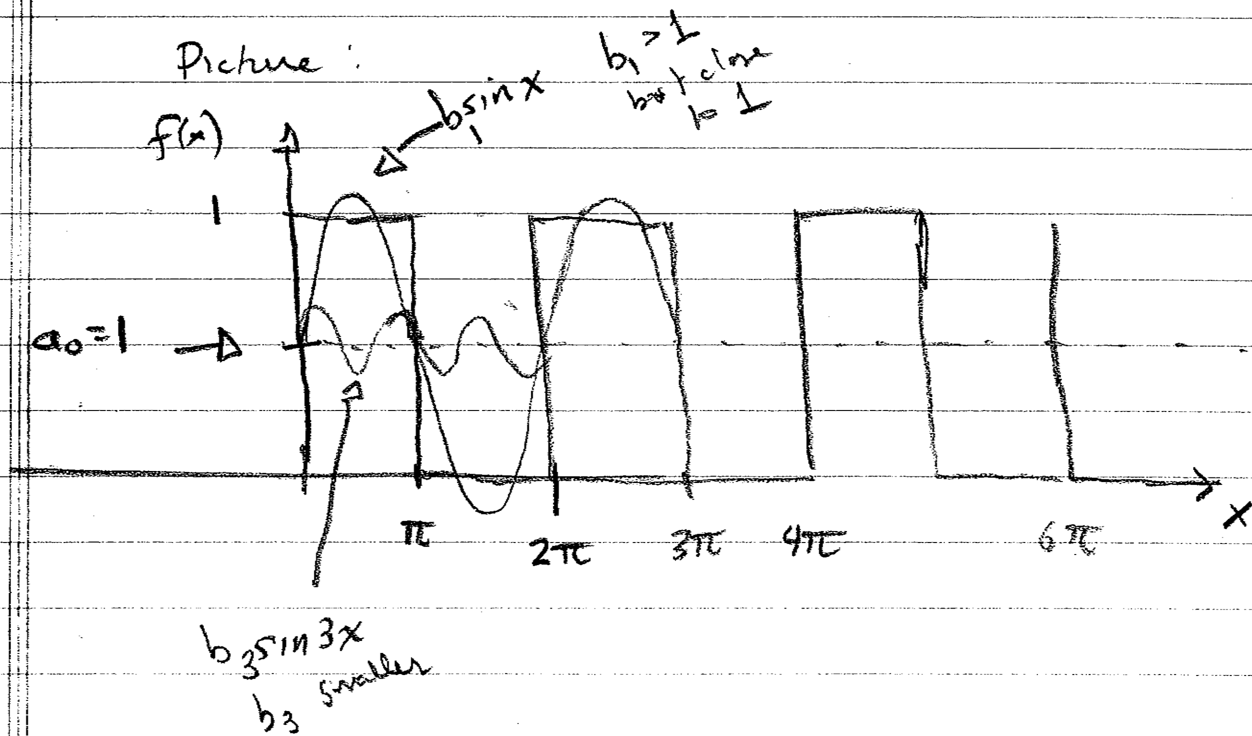
$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

(Longer / shorter periods can be handled via use of $\cos n2\pi x/T$ and $\sin n2\pi x/T$.)

J1

Conditions on f? Remarkably unrestrictive. f can be discontinuous! In fact, must merely have a finite # of (finite) discontinuities), Also finite # of maxima and minima. We'll see something a bit bizarre, though about behavior near discontinuities (Gibbs).

Picture:



(II)'

This seems kind of odd! Why should all (periodic) functions be expandable in sines and cosines?!

But really no less odd than familiar Taylor expansion

$$f(x) = f(0) + f'(0)x + f''(0)x^2/2 + \ldots$$

$$\left.\begin{aligned} e^x &= 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \ldots \\ \sin x &= x - x^3/6 + x^5/120 - \ldots \\ \cos x &= 1 - x^2/2 + x^4/24 \\ \ln(1+x) &= x - x^2/2 + x^3/3 \ldots \end{aligned}\right\}$$

any bizarre function can be expanded in $1, x, x^2, \ldots$

actually expanding about 1 not $\emptyset$ because $\ln x$ singular at $x = 0$

$\boxed{\text{Coefficients}}$

$$f(x) = a_0/2 + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx$$

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx\, dx \qquad \text{use } [0, L]$$

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx\, dx$$

Why?

Go back to vectors

$$|\vec{v}\rangle = \sum' v_i |e_i\rangle$$

why is $v_j = \langle e_j | v \rangle$?

Because of orthonormality

$$\langle e_j | v \rangle = \sum v_i \underbrace{\langle e_j | e_i \rangle}_{\delta_{ij}} = v_j$$

Here $\cos mx$ and $\sin mx$ are orthogonal because they are eigenvectors of a self adjoint (Hermitian) differential operator.

$$\int_0^{2\pi} dx \cos nx \cos mx = \delta_{nm} \quad (\text{normalization})$$

$$\int_0^{2\pi} dx \sin nx \sin mx = \delta_{nm} \quad (\quad " \quad)$$

$$\int_0^{2\pi} dx \cos nx \sin mx = 0$$

Normalization factor

$$\frac{1}{T}\int_T \cos^2 \omega t \, dt = 1/2$$

$$1/T \int_T \sin^2 \omega t \, dt = 1/2$$

"Average of $\cos^2$ and $\sin^2$ over full period is 1/2"

Why? $\cos^2 \omega t + \sin^2 \omega t = \underline{1}$

and over full period $\cos^2$ and $\sin^2$ are identical.

$\to$ (normalization) is π

$$1/\pi \int_0^{2\pi} dx \cos nx \cos mx = \delta_{nm}$$

and similarly for sin.

Don't like all this Hermitian differential operator mumbo-jumbo? Can prove directly

$$\int_0^{2\pi} \cos nx \cos mx \, dx = 1/2 \int_0^{2\pi} \left[\cos(n+m)x + \cos(n-m)x\right] dx$$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$= 0$

unless $n = m$

$\to \pi$

C-3

To beat a dead horse, then

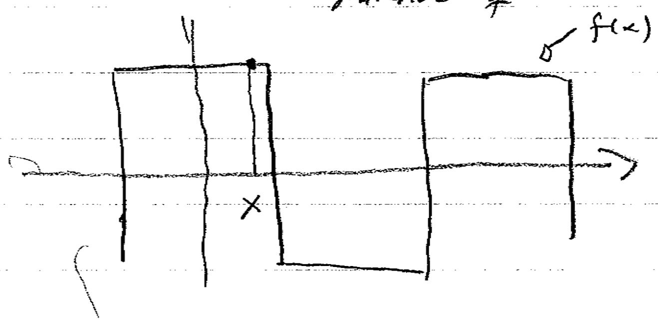
$$f(x) = a_0/2 + \sum a_n \cos nx + \sum b_m \cos mx$$

$$\int_0^{2\pi} \cos mx \, f(x)\, dx = a_m \pi$$ (all other integrals vanish)

Gibbs Phenomena

One really nice thing about Fourier series is that they can expand functions with discontinuities. But all is not perfect.

Consider Fourier expansion of



What is true is that if one sits at any position x and includes more and more terms one can always make

$$\left| f(x) - \sum_{1}^{N} a_n \cos nx \right| < \epsilon \quad \text{any } \epsilon.$$

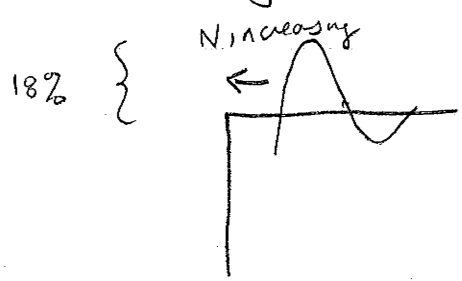
Convergence

but what one cannot do is find an N such that

$$\left| f(x) - \sum_{1}^{N} a_n \cos nx \right| < \epsilon \qquad \forall x$$

uniform convergence

There is an overshoot by about 18% which moves steadily towards discontinuity point, leaving behind it points which are converging



Can already see role symmetry might play in Fourier expansion.

Why might this work? Maybe not able to prove rigorously right now but one argument is that sin nx and cos nx are eigenfunctions of the operator d^2/dx^2. That is

$$d^2y/dx^2 = -n^2 y$$

is obeyed. Then, as for matrices, this set is orthonormal and forms a basis.

Well, for matrices, there was a condition: M had to be Hermitian.

Let's present the argument more carefully...

DF-1

Discrete Fourier Transform

$$f(x) = a_0/2 + \sum_1^{\infty} a_n \cos 2\pi n x/L + \sum_1^{\infty} b_n \sin 2\pi n x/L$$

$$a_n = 2/L \int_0^L \cos 2\pi n x/L \, f(x)\, dx$$

$$b_n = 2/L \int_0^L \sin 2\pi n x/L \, f(x)\, dx$$

$$= c_0 + \sum_1^{\infty} c_n e^{i 2\pi n x/L} + \sum_{-1}^{-\infty} c_n e^{i 2\pi n x/L}$$

with $c_0 = a_0/2 \qquad c_n = a_n/2 + b_n/2i \qquad n > 0$

$$= a_{-n}/2 - b_n/2i \qquad n < 0$$

$n > 0 \qquad c_n = \frac{1}{2}(a_n - i b_n) = 1/L \int_0^L \left(\cos \frac{2\pi n x}{L} - i \sin \frac{2\pi n x}{L}\right) f(x)\, dx$

$$c_n = \frac{1}{L} \int_0^L e^{-inx\frac{2\pi}{L}} f(x)\, dx$$

$n < 0 \qquad c_n = 1/2(a_{-n} + i b_{-n}) = \frac{1}{L} \int_0^L \left(\cos \frac{2\pi(-n)x}{L} + i \sin \frac{2\pi(-n)x}{L}\right) f(x)\, dx$

$$= 1/L \int_0^L e^{-inx\frac{2\pi}{L}} f(x)\, dx \qquad \text{also!}$$

$$c_0 = a_0/2 = 1/L \int_0^L f(x)\, dx$$

Same formula for all c_n!

$$\boxed{f(x) = \sum_{-\infty}^{\infty} c_n e^{i 2\pi n x/L} \qquad c_n = 1/L \int_0^L e^{-inx\, 2\pi/L} f(x)\, dx}$$

DF - 2

$$\frac{1}{L}\int_0^L e^{\frac{2\pi i n x}{L}} e^{-\frac{2\pi i n' x}{L}}\, dx = \delta_{nn'}$$

proof $= \frac{1}{L}\int_0^L \left(\cos\frac{2\pi n x}{L} + i\sin\frac{2\pi n x}{L}\right)\left(\cos\frac{2\pi n' x}{L} - i\sin\frac{2\pi n' x}{L}\right) dx$

$$= \frac{1}{L}\left[\delta_{nn'}\frac{1}{2} + \delta_{nn'}\frac{1}{2}\right] L$$

$\square$ formula on p DF-1 slightly strange!

one variable discrete n

one variable continuous x

More symmetric constructions frequently occur

DO FI $\rightarrow$ Second, both discrete

Suppose instead of $f(x)$ continuous on $[0, L]$ you want to represent L discrete quantities f_ℓ

$\ell = 1, 2, 3, \ldots L$

DF-3

$f(x)$ ← continuous

f_ℓ ← $\ell = 1, 2 \ldots L$ <u>discrete</u> values

$$f_\ell = 1/\sqrt{L} \sum_{n=1}^{L} c_n e^{i 2\pi n \ell / L}$$

$$c_n = 1/\sqrt{L} \sum_{\ell=1}^{L} f_\ell \, e^{-i 2\pi n \ell / L}$$

sometimes "k_n" $\equiv 2\pi n / L$

$$f_\ell = 1/\sqrt{L} \sum_{k} c_k e^{ik\ell}$$

often $\frac{1}{L} \sum_{\ell=1}^{L} e^{2\pi i (n-n')\ell / L} = \delta_{n,n'}$

$$c_k = \frac{1}{\sqrt{L}} \sum_{\ell} f_\ell \, e^{-ik\ell}$$

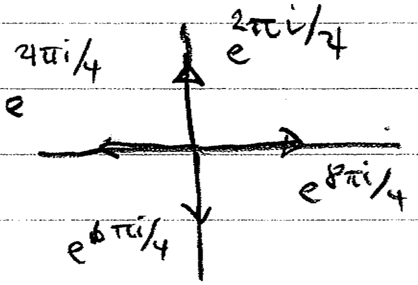
orthogonality $\quad 1/L \sum_{\ell=1}^{L} e^{2\pi i m \ell / L} = \delta_{m,0}$ $\quad$ actually $= 1$ if m = multiple of L.

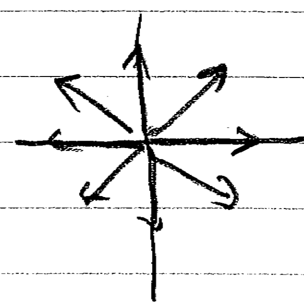
obviously true for $m = 0$ $(L, 2L, \ldots)$

Easy geometric interpretation for $m = 1$

$$e^{2\pi i \ell / L} = L^{th} \text{ roots of } 1$$

L = 4

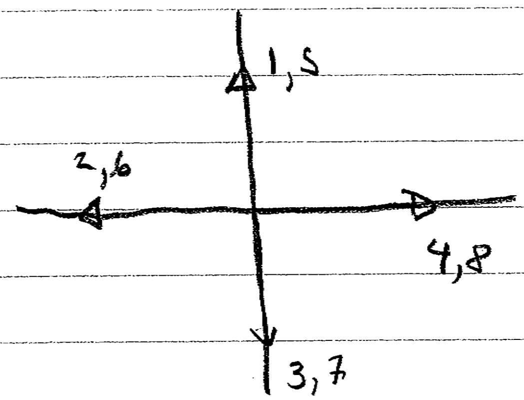




L = 8

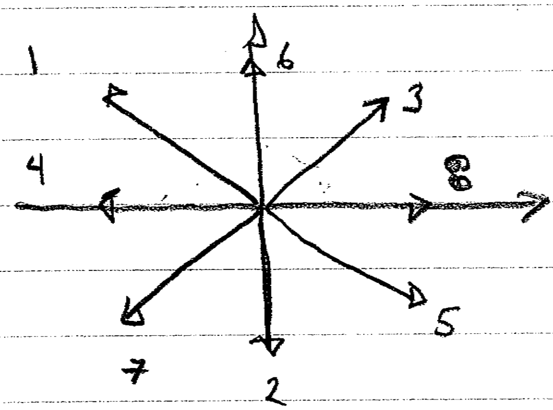
$\sum = 0$ obviously!

What about $m=2$? $m=2 \quad L=8$



$\Sigma = 0$

$m=3 \quad L=8$



The L^{th} roots of 1 add up to zero

Any power of the L^{th} roots of 1 add up to zero

unless that power is an integer multiple of L.

Application of orthogonality

Given $\{f_\ell\}$ define

$$c_n = 1/\sqrt{L} \cdot \sum_{\ell=1}^{L} f_\ell \, e^{-2\pi i n \ell/L}$$

$$\sum_{n=1}^{N} e^{+i2\pi \ell' n/L} c_n = 1/\sqrt{L} \sum_{n=1}^{N} e^{+i2\pi \ell' n/L} \sum_{\ell} f_\ell e^{-i2\pi n \ell/L}$$

$$= \sqrt{L} \; f_{\ell'}$$

$$\therefore \quad f_{\ell'} = \frac{1}{\sqrt{L}} \sum_{n} e^{i2\pi n \ell'/L} c_n$$

Actually we saw all this before!

Matrix $L \times L$

$$\begin{bmatrix} A & B & 0 & 0 & 0 & B \\ B & A & B & 0 & 0 & 0 \\ 0 & B & A & B & 0 & 0 \\ & & & & & \\ & & & & & \end{bmatrix}$$

Eigenvectors $|v_n\rangle$ have components $e^{2\pi i n \ell/L} \frac{1}{\sqrt{L}}$

$$\delta_{n,n'} = \langle v_{n'} | v_n \rangle = \frac{1}{L} \sum_{\ell} e^{-2\pi i n' \ell/L} e^{+i2\pi i n \ell/L}$$

components