What is it? Expand any $f(x)$ which has period $2\pi$:

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

(Larger/shoulder periods can be handled via use of $\cos n2\pi x/T$ and $\sin n2\pi x/T$)

Conditions on $f$? Remarkably unrestrictive:

$f$ can be discontinuous! In fact, must merely have a finite # of (finite) discontinuities.

Also finite # of maxima and minima, we'll see something a bit bizarre though about behavior near discontinuities (Gibbs).

Picture:

- $a_0 = 1$
- $b_3 \approx \frac{3}{\pi}$
- $b_5 \approx \frac{5}{3\pi}$

Graph showing oscillations and behavior near discontinuities.
This seems kind of odd! Why should all (periodic) functions be expandable in sines and cosines?

But really no less odd than familiar Taylor expansions:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots$$

Any binomial function can be expanded in $1, x, x^2, \ldots$ actually expanding about 1 not 0

because $\tan x$ singular at $x = 0$
\[ f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \]

\[ a_n = \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx \quad \text{where} \quad [0, \pi] \]

\[ b_n = \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx \]

Why?

Go back to vectors

\[ |v\rangle = \sum v_i |e_i\rangle \]

Why is \( v_i = \langle e_i | v \rangle \)?

Because of orthonormality\( \delta_{ij} \)

\[ \langle e_j | v \rangle = \sum v_i \langle e_j | e_i \rangle = v_j \]

Here, \( \cos nx \) and \( \sin mx \) are orthogonal because they are eigenvectors of a self-adjoint (Hermitian) differential operator.

\[ \int_{0}^{\pi} dx \cos nx \cos mx = \delta_{nm} \quad (\text{normalization}) \]

\[ \int_{0}^{\pi} dx \sin nx \sin mx = \delta_{nm} \quad (\text{orthogonality}) \]

\[ \int_{0}^{\pi} dx \cos nx \sin mx = 0 \]
Normalization factor:

$$\frac{1}{T} \int_{0}^{T} \cos^2 \omega t \, dt = \frac{1}{2}$$  "Average of \( \cos^2 \) and \( \sin^2 \) over full period is \( \frac{1}{2} \)"

$$\frac{1}{T} \int_{0}^{T} \sin^2 \omega t \, dt = \frac{1}{2}$$

Why? \( \cos^2 t + \sin^2 t = 1 \)

and over full period \( \cos^2 \) and \( \sin^2 \) are identical.

\[ \Rightarrow (\text{normalized}) \text{ is} \]

$$\frac{1}{\pi} \int_{0}^{\pi} \cos nx \cos mx \, dx = \delta_{mn}$$

and similarly for \( \sin \).

Don't like all this Hermitian jibberish? Can prove directly:

$$\frac{1}{\pi} \int_{0}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2} \int_{0}^{\pi} \left[ \cos (n+m)x + \cos (n-m)x \right] \, dx$$

$$\cos (A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos (A-B) = \cos A \cos B + \sin A \sin B$$

when \( n = m \)

$$= 0$$

$$\pi$$
To beat a dead house, then

\[ f(x) = a_0/2 + \sum a_n \cos nx + \sum b_m \sin mx \]

\[ \int_0^{2\pi} \cos(mx) f(x) \, dx = \begin{cases} \frac{\pi}{m} & \text{(all other integrals vanish)} \\ \end{cases} \]
One really nice thing about Fourier series is that they can expand functions with discontinuities, but all is not perfect.

Consider Fourier expansion of \( f(x) \):

\[
\text{What is true is that if one \( \sum \) at any position \( x \) and includes more and more terms one can always make
}
\[
| f(x) - \sum_{n=1}^{\infty} a_n \cos nx | < \epsilon \quad \text{for any } \epsilon. \quad \text{Convergence}
\]

but what one cannot do is find an \( N \) such that
\[
| f(x) - \sum_{n=1}^{N} a_n \cos nx | < \epsilon \quad \forall x \quad \text{uniform convergence}
\]

There is an overshoot by about 18% which moves steadily towards discontinuity point, leaving behind it point which are concave.

\[
18\% \{ \text{Overshoot} \}
\]
can already see role of symmetry might play
in Fourier expansion.

Why might this work? Maybe not able
to prove rigorously right now, but one argument is
that sin nx and cos nx are eigenfunctions of the
operator \( \frac{d^2}{dx^2} \). That is

\[
\frac{d^2 y}{dx^2} = -n^2 y
\]

is obeyed. Then, as for matrices, this
set is orthonormal and forms a basis.

Well, for matrices, there was a condition!
M had to be Hermitian.

Let's present the argument more carefully...
Discrete Fourier Transform

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi nx}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{2\pi nx}{L} \]

\[ a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{2\pi nx}{L} \, dx \]

\[ b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{2\pi nx}{L} \, dx \]

\[ \sum_{n=0}^{\infty} c_n e^{i \frac{2\pi nx}{L}} + \sum_{n=0}^{\infty} d_n e^{-i \frac{2\pi nx}{L}} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( c_n - i d_n \frac{1}{2} \right) e^{i \frac{2\pi nx}{L}} + \sum_{n=1}^{\infty} \left( c_n + i d_n \frac{1}{2} \right) e^{-i \frac{2\pi nx}{L}} \]

with \( a_0 = a_0 / 2 \), \( c_n = a_n / 2 + i b_n / 2 \), \( n > 0 \)

\( = a_0 / 2 - i b_n / 2 \), \( n < 0 \)

\( \int_{0}^{L} e^{-i \frac{2\pi nx}{L}} f(x) \, dx \)

\( \int_{0}^{L} e^{i \frac{2\pi nx}{L}} f(x) \, dx \)

\( c_n = \frac{1}{L} \int_{0}^{L} \left( \cos \frac{2\pi nx}{L} + i \sin \frac{2\pi nx}{L} \right) f(x) \, dx \)

\( c_n = \frac{1}{L} \int_{0}^{L} e^{i \frac{2\pi nx}{L}} f(x) \, dx \)

\( c_n = \frac{1}{L} \int_{0}^{L} e^{-i \frac{2\pi nx}{L}} f(x) \, dx \)

Same formula for all \( c_n \) it

\[ f(x) = \sum_{n=0}^{\infty} c_n e^{i \frac{2\pi nx}{L}} \]

\[ c_n = \frac{1}{L} \int_{0}^{L} e^{-i \frac{2\pi nx}{L}} f(x) \, dx \]
\[ \frac{1}{L} \int_0^L e^{\frac{-2\pi n_1 x}{L}} e^{\frac{2\pi n_1 x}{L}} \, dx = \delta_{nn_1} \]

\[ \text{proof} = \frac{1}{L} \int_0^L \left( \cos \frac{2\pi n_1 x}{L} + i \sin \frac{2\pi n_1 x}{L} \right) \left( \cos \frac{2\pi n_1 x}{L} - i \sin \frac{2\pi n_1 x}{L} \right) \, dx \]

\[ = \frac{1}{L} \left[ \delta_{nn_1} \frac{L}{2} + \delta_{nn_1} \frac{L}{2} \right] \]

- Formula a p DF-1 slightly change
- one variable denote \( L \)
- one variable continuous \( x \)

More symmetric constructions frequently occur

Second, both denote

Suppose instead of \( f(x) \) continuous on \([0, L]\)
you want to represent L discrete quantities \( f_L \)
\[ L = 1, 2, 3, \ldots \]
\[ f(x) \text{ (continuous)} \]

\[ f_\alpha = \frac{1}{\sqrt{L}} \sum_{n=1}^{L} c_n e^{i2\pi n \alpha / L} \]

\[ c_n = \frac{1}{\sqrt{L}} \sum_{\alpha=1}^{L} f_\alpha e^{-i2\pi n \alpha / L} \]

Sometimes \( \frac{k}{n} \equiv \frac{2\pi n}{L} \)

\[ f_\alpha = \frac{1}{\sqrt{L}} \sum_{k} c_k e^{i k \alpha} \]

\[ L \sum_{\alpha=1}^{L} e^{i2\pi m \alpha / L} = \delta_{m, 0} \]

orthogonality

Easily geometric interpretation for \( m = 1 \)

\[ e^{\frac{2\pi i}{L}} = \text{rotation of } 1 \]

\[ L = 4, e^{\frac{2\pi i}{4}}, e^{\frac{2\pi i}{4}} \]

\[ L = 8 \]

\[ \sum = 0 \text{ obviously} \]
What about $m = 2$? $m = 2 \quad \Sigma = \varnothing$

$m = 3 \quad \lambda = \varnothing$

The $L$th root of $I$ add up to zero.

Any power of the $L$th root of $I$ add up to zero.

unless that power is an integer multiple of $L$. 
Application of orthogonality

Given \( \{f_e\} \) define

\[
e_n = \frac{1}{\sqrt{L}} \sum_{e=1}^{L} f_e e^{-\frac{2\pi im}{L}}
\]

\[
\sum_{n=1}^{N} e^{\frac{i2\pi n m}{L}} c_n = \frac{1}{\sqrt{L}} \sum_{e=1}^{L} f_e e^{-\frac{2\pi im}{L}} c_e
\]

\[
= \sqrt{L} f_e !
\]

\[
i.e. \quad f_e = \frac{1}{\sqrt{L}} \sum_{n} e^{\frac{i2\pi mn}{L}} c_n
\]

Actually we saw all this before!

Matrix

\[
\begin{pmatrix}
A & B & 0 & 0 & 0 & B \\
B & A & B & 0 & 0 & 0 \\
0 & B & A & B & 0 & 0
\end{pmatrix}
\]

Eigenvalues \( \{ \lambda_n \} \) have components

\[
e^{\frac{2\pi in}{L}} \frac{1}{\sqrt{L}}
\]

\[
\delta_{nn'} = \langle \lambda_n, \lambda_{n'} \rangle = \frac{1}{\sqrt{L}} \sum_{e=1}^{L} e^{-\frac{2\pi im}{L}} e^{\frac{2\pi im}{L}}
\]