FI-1

Fourier integrals

DO BACK PAGE FIRST

More symmetric : Both continuous

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; c(k)\, e^{ikx}$$

$$c(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \; f(x)\, e^{-ikx}$$

Can regard these as the $L \to \infty$ limit

$$e^{i \frac{2\pi n}{L} x}$$

$$\ldots, -\frac{4\pi}{L}, -\frac{2\pi}{L}, 0, +\frac{2\pi}{L}, +\frac{4\pi}{L}, \ldots$$

$L \to \infty \implies$ continuous

Bonus: $f(x)$ unrestricted now!!

Normalization ?

Tied up with proof of inversion

$$f(x) = \int_{-\infty}^{\infty} dk \; c(k)\, e^{ikx}$$

$$\int dx \; e^{-ik'x} f(x) = \int dx \, e^{-ik'x} \int dk \; c(k)\, e^{ikx}$$

$$= \int dk \; c(k) \underbrace{\int e^{i(k-k')x} dx}_{\delta(k-k')}$$

$$= c(k')$$

FI-12

we know

$$\frac{1}{L}\int_{-L/2}^{L/2} e^{i\left(\frac{2\pi}{L}n\right)x}\, e^{-i\left(\frac{2\pi n'}{L}\right)x}\, dx = \delta_{n,n'}$$

as $n \to \infty$ $\quad \frac{2\pi}{L} n \to$ continuous variable k

$$\delta_{n,n'} \to \delta(k-k')$$

what about normalization?

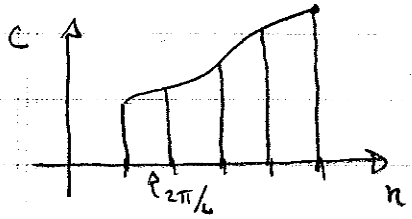
Different ways to look at it

(1) $\quad f(x) = \sum_n c_n e^{i 2\pi n x/L}$

$c_n = \frac{1}{L}\int_0^L e^{-inx\,2\pi/L} f(x)\,dx$

Division of coefficients into 1 and 1/L arbitrary. Only rule is product is 1/L

$\sum_n c_n \Rightarrow \frac{L}{2\pi}\int dk\, c(k)$ (usual prescription for converting sum to integral)



$\frac{1}{L}$ and $\frac{L}{2\pi}$ are coefficients

product is $1/2\pi$

$\sum c_n \frac{2\pi}{L} \Rightarrow \int dk\, c(k)$

L large

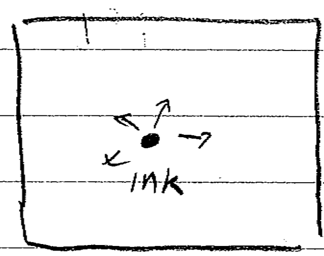
$\to$ $\quad f(x) = 1/\sqrt{2\pi}\int dk\, c(k)\, e^{ikx}$

$c(k) = 1/\sqrt{2\pi}\int dx\, f(x)\, e^{-ikx}$

$$\frac{1}{2\pi}\int e^{ikx}\,dx = \delta(k)$$

FI-3

Application: Diffusion Eqn (⟹ Schr. Eqn)



1-d

$\rho(x,t)$

$$d\rho/dt = D\, \partial^2 \rho/\partial x^2$$

diffusion constant

$$\rho(x,t) = f(x) g(t)$$

$$f(x) \frac{\partial g}{dt}(t) = D\, g(t) \frac{d^2 f(x)}{dx^2}$$

$$\frac{1}{g(t)} \frac{dg}{dt} = \frac{D}{f(x)} \frac{d^2 f}{dx^2} \equiv -Dk^2$$

$$f(x) = e^{ikx}$$

$$g(t) = e^{-Dk^2 t}$$

$e^{ikx} e^{-Dk^2 t}$ is soln for any k!

general soln is combination $\quad \rho(x,t) = \int c(k) e^{ikx} e^{-Dk^2 t} dk$

$$\rho(x,0) = \int c(k) e^{ikx} dk$$

$$\rightarrow c(k) = \frac{1}{2\pi} \int \rho(x,0) e^{-ikx} dx$$

FI-4

$$\boxed{\begin{aligned} p(x,t) &= \int c(k)\, e^{ikx}\, e^{-Dk^2 t}\, dk \\ c(k) &= \frac{1}{2\pi} \int p(x,0)\, e^{-ikx}\, dx \end{aligned}}$$

Eg $\quad p(x,0) = \delta(x) \qquad c(k) = 1/2\pi$

$$p(x,t) = 1/2\pi \int e^{ikx}\, e^{-Dk^2 t}\, dk$$

$$= \frac{1}{2\pi} \int e^{-Dt(k^2 - ikx/Dt)}\, dk$$

$$= \frac{1}{2\pi} \int e^{-Dt(k - ix/2Dt)^2}\, e^{-x^2/4Dt}\, dk$$

$$= \frac{1}{2\pi} \sqrt{\frac{\pi}{Dt}}\, e^{-x^2/4Dt}$$

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}}\, e^{-x^2/4Dt}$$

$\langle x^2 \rangle \sim Dt \qquad$ "diffusion result"

$\cancel{\int dx\, p(x,0) = \int c(k)\, dk}$

$$\int dx\, p(x,t) = \int c(k) \underbrace{\int dx\, e^{ikx}}_{\delta(k)} e^{-Dk^2 t}\, dk$$

$= 2\pi \int dK\, c(k) \quad \leftarrow$ time independent

$\cancel{\left[= \int dk \int dx\, p(x,0)\, e^{-ikx} \right.}$

$\cancel{\left. = 2\pi \int dx\, p(x,0) \right]}$

Useful to put together (eliminate $c(k)$)

$$p(x,t) = \int dk\, e^{ikx}\, e^{-Dk^2t} \int p(x',0)\, e^{-ikx'} \frac{dx'}{2\pi}$$

$$= \int dx'\, p(x',0) \int \frac{dk}{2\pi}\, e^{-Dk^2t}\, e^{ik(x-x')}$$

We just did this integral

$$p(x,t) = \int dx'\, p(x',0) \underbrace{\frac{1}{\sqrt{4\pi Dt}}\, e^{-(x-x')^2/4Dt}}$$

What is this called?

- Green's function
- propagator

} $G(x-x',t)$

How much function $p(x',0)$ spreads to x in time t.

Repeat

$$p(x',0) = \delta(x')$$

$$p(x,t) = G(x,t)$$

$G(x,t)$ is soln of diff eqn starting with δ function initial condition