3.1. ZERO-TEMPERATURE THEORY: GENERAL FORMALISM

It is convenient, for the type of perturbation theory to be developed, to transform state vectors and operators from the Heisenberg to the *interaction picture.* This is an intermediate form between the Schrödinger and Heisenberg pictures, in which both the state vectors and the operators are time-dependent but in which the operators have a particularly simple time dependence determined by the unperturbed hamiltonian. With $H = H_0 + H_1$, the state vector in the interaction picture is defined by

$$|\Psi_I(t)\rangle = e^{iH_0 t}|\Psi_S(t)\rangle, \tag{3.1.1}$$

where $|\Psi_S\rangle$is the Schrödinger state vector, and satisfies

$$i\frac{\partial|\Psi_I\rangle}{\partial t} = H_1(t)|\Psi_I\rangle, \tag{3.1.2}$$

where

$$H_1(t) = e^{iH_0 t}H_1 e^{-iH_0 t}. \tag{3.1.3}$$

(The origin of time has been chosen such that the Heisenberg, interaction and Schrödinger pictures all coincide at $t = 0$.) In general, operators in the interaction picture are related to Schrödinger operators by

$$\mathcal{O}_I(t) = e^{iH_0 t}\mathcal{O}_S e^{-iH_0 t}. \tag{3.1.4}$$

We now define the *time development operator* $U(t, t')$ by

$$|\Psi_I(t)\rangle = U(t, t')|\Psi_I(t')\rangle, \tag{3.1.5}$$

and it follows from the time variation (3.1.1) and (1.4.2) of the state vectors that

$$U(t, t') = e^{iH_0 t}\, e^{-iH(t-t')}\, e^{-iH_0 t'}. \tag{3.1.6}$$

This expression cannot in general be simplified further since H and H_0 do not commute. The operator U obviously satisfies the Schrödinger equation

$$i\frac{\partial U(t, t')}{\partial t} = H_1(t)U(t, t'), \tag{3.1.7}$$

and possesses the *group property*

$$U(t, t')U(t', t'') = U(t, t'') \tag{3.1.8}$$

and the properties of *hermiticity*

$$U(t, t') = U^{\dagger}(t', t) \tag{3.1.9}$$

and *unitarity*

$$U(t, t')U^{\dagger}(t, t') = 1. \tag{3.1.10}$$

State vectors $|\Psi_H\rangle$ and operators $\mathcal{O}_H(t)$ in the Heisenberg picture are related to those in the interaction picture by

$$|\Psi_H\rangle = e^{iHt}|\Psi_S(t)\rangle = e^{iHt}\, e^{-iH_0 t}|\Psi_I(t)\rangle = U^{\dagger}(t, 0)|\Psi_I(t)\rangle, \tag{3.1.11}$$

and

$$\mathcal{O}_H(t) = U^{\dagger}(t, 0)\mathcal{O}_I(t)U(t, 0). \tag{3.1.12}$$

We now introduce the *adiabatic hypothesis*. We wish to express the Green's function in terms of expectation values with respect to the ground state $|\Phi_0\rangle$ of the unperturbed hamiltonian H_0. We assume that at time $t = -\infty$ the system is in the state $|\Phi_0\rangle$, that the interaction is then "switched on" to increase infinitely slowly so as to reach its full value H_1 at time $t = 0$, and is then switched off again to vanish adiabatically at $t = +\infty$. To formulate this time variation mathematically we multiply H_1 by a factor $\exp(-\epsilon|t|)$ and allow ϵ to tend to zero at the end of the calculations. This somewhat artificial procedure ensures that all the integrals appearing in the formalism have well-defined values, and one can show [Gell-Mann and Low (1951)] that the quantity

$$\frac{U(0, -\infty)|\Phi_0\rangle}{\langle\Phi_0|U(0, -\infty)|\Phi_0\rangle} \tag{3.1.13}$$

obtained in this way at time $t = 0$ exists in the limit $\epsilon = 0$ and is an eigenstate of the Hamiltonian $H_0 + H_1$. [The argument has to be formulated with care, because the numerator and denominator of (3.1.13) are separately infinite in the limit $\epsilon = 0$.] The theorem defines the state that develops from the non-interacting hamiltonian by adiabatic switching on of the interaction, but it does not require this state to be the ground state of the interacting system. To obtain the new ground state by this procedure it is necessary to assume in addition that the eigenstates of the perturbed system evolve from the unperturbed eigenstates without any crossing of states. This excludes the possibility of instabilities such as superconductivity which lead to a new kind of ground state with a different symmetry and a lower energy than the state obtained adiabatic-

ally from $|\Phi_0\rangle$. In fact the ground state energy in this case does not have a perturbation series in the coupling constant. In Chap. 10 we shall discuss how the Green's function approach can be adapted to deal with such phenomena.

We now use this formalism to re-express the Green's function in a form in which the expectation values refer to the unperturbed ground state $|\Phi_0\rangle$. The state vector used in the original definition (1.4.6) of G is $U(0, -\infty)|\Phi_0\rangle$, and we have

$$U(0, -\infty)|\Phi_0\rangle = U(0, \infty)U(\infty, -\infty)|\Phi_0\rangle = U(0, \infty)S|\Phi_0\rangle, \qquad (3.1.14)$$

where we have introduced the *S-matrix* $S = U(\infty, -\infty)$. Since the interaction is again zero at $t = +\infty$, the state $S|\Phi_0\rangle$ can differ from $|\Phi_0\rangle$ only by a phase factor of unit modulus, *provided* that the state which develops from $|\Phi_0\rangle$ is non-degenerate. We make this assumption and thus have

$$S|\Phi_0\rangle = e^{i\alpha}|\Phi_0\rangle. \qquad (3.1.15)$$

Transforming the Heisenberg operators in the Green's function (1.4.6) to the interaction picture by means of Eq. (3.1.12) and using the properties (3.1.8) and (3.1.9) of the operator U we have, for $t > 0$,[1]

$$\begin{aligned} G_{ij}(t) &= -i\langle\Phi_0|S^\dagger U^\dagger(0, \infty)U^\dagger(t, 0)u_i(t)U(t, 0)u_j(0)U(0, -\infty)|\Phi_0\rangle \\ &= -i\, e^{-i\alpha}\langle\Phi_0|U(\infty, t)u_i(t)U(t, 0)u_j(0)U(0, -\infty)|\Phi_0\rangle \\ &= -i\, e^{-i\alpha}\langle\Phi_0|T[u_i(t)u_j(0)U(\infty, t)U(t, 0)U(0, -\infty)]|\Phi_0\rangle \\ &= -i\, e^{-i\alpha}\langle\Phi_0|T[u_i(t)u_j(0)S]|\Phi_0\rangle. \end{aligned}$$

Also, using Eq. (3.1.15),

$$\langle\Phi_0|S|\Phi_0\rangle = e^{i\alpha}\langle\Phi_0|\Phi_0\rangle = e^{i\alpha}.$$

The same expression is obtained for G_{ij} when $t < 0$, and we thus have the result

$$G_{ij}(t) = -i\,\frac{\langle\Phi_0|T[u_i(t)u_j(0)S]|\Phi_0\rangle}{\langle\Phi_0|S|\Phi_0\rangle}. \qquad (3.1.16)$$

Thus, by use of the time-ordering symbol, we have obtained a compact form for the Green's function in the interaction picture. Note that the T symbol applies to the operators in S as well as to the product $u_i(t)u_j(0)$.

[1] This transformation can again be justified by an $\epsilon \to 0$ limiting procedure; see Fetter and Walecka (1971), p. 83. In the rest of this section all operators are in the interaction picture, and we omit the suffix I.

The use of the T operator here is a generalization of its use in Chap. 1. It is to be understood to act in terms of the time labels of the $u(t)$ operators implicit in the expansion of S [see Eq. (3.1.23)]. The idea is that, in any given term in the expansion, the T operator automatically orders all operators $u(t_1)u(t_2)\ldots u(t_n)$ in such a term in order of occurrence of the t_i labels. Thus, if $t_{i_1} > t_{i_2} > \ldots > t_{in}$, then

$$T\{u(t_1)u(t_2)\ldots u(t_n)\} = u(t_{i_1})u(t_{i_2})\ldots u(t_{in}). \tag{3.1.17}$$

The form (3.1.16) has the advantage that the interaction H_1 is contained entirely in S, so that we can confine attention to the perturbation series for this quantity. To derive this, standard field-theoretic methods are available.

To obtain the required series we note that the differential equation (3.1.7) together with the initial condition $U(t, t) = 1$ is equivalent to the integral equation

$$U(t, t') = 1 - i\lambda \int_{t'}^{t} dt_1 H_1(t_1)U(t_1, t'). \tag{3.1.18}$$

Here we have again introduced a coupling constant λ into the interaction. We solve Eq. (3.1.18) by direct iteration, and obtain

$$U(t, t') = 1 - i\lambda \int_{t'}^{t} dt_1 H_1(t_1) + (i\lambda)^2 \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 H_1(t_1)H_1(t_2)$$
$$+ \cdots + (-i\lambda)^n \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \ldots \int_{t'}^{t_{n-1}} dt_n H_1(t_1)\ldots H_1(t_n) + \cdots. \tag{3.1.19}$$

The order of the factors is important in these integrals, since $H_1(t_1)$ and $H_1(t_2)$ do not commute if $t_1 \neq t_2$. The essential step in the Feynman–Dyson evaluation of the terms in this series now comes in a trick, explicitly formulated by Dyson, in which the "time-labeling" of the operators is removed in favor of the effect of the time-ordering operator T. Consider, in second order,

$$\int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 H_1(t_1)H_1(t_2) + \int_{t'}^{t} dt_1 \int_{t_1}^{t} dt_2 H_1(t_2)H_1(t_1). \tag{3.1.20}$$

Interchange of the order of integration in the second term shows that the two terms of (3.1.20) are equal. In the first term $t_1 \geqslant t_2$; in the second term $t_2 \geqslant t_1$. Hence, using the T symbol, we can write the first term alone as

$$\tfrac{1}{2} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 T[H_1(t_1)H_1(t_2)].$$

This transformation can be extended to the general case of n arguments, when there are $n!$ equal terms of the type (3.1.20), corresponding to the $n!$ possible orderings of the time labels $t_1, \ldots, t_n$. The new form of the series (3.1.19) is thus

$$\begin{aligned} U(t, t') = 1 - i\lambda \int_{t'}^{t} dt_1 H_1(t_1) + \frac{(i\lambda)^2}{2!} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 T[H_1(t_1)H_1(t_2)] \\ + \cdots + \frac{(-i\lambda)^n}{n!} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \ldots \int_{t'}^{t} dt_n T[H_1(t_1) \ldots H_1(t_n)] \\ + \cdots . \end{aligned} \tag{3.1.21}$$

This can be summed symbolically as an exponential series to give

$$U(t, t') = T \exp\left\{-i\lambda \int_{t'}^{t} dt_1 H_1(t_1)\right\}; \tag{3.1.22}$$

in practice, however, this expression must always be interpreted as the series (3.1.21). Thus S has the expansion

$$\begin{aligned} S = 1 - i\lambda \int_{-\infty}^{\infty} dt_1 H_1(t_1) + \frac{(i\lambda)^2}{2!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 T[H_1(t_1)H_1(t_2)] \\ + \cdots + \frac{(-i\lambda)^n}{n!} \int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_n T[H_1(t_1) \ldots H_1(t_n)] + \cdots, \end{aligned} \tag{3.1.23}$$

and this gives for the Green's function (3.1.16) the series

$$\begin{aligned} G_{ij}(t) = G^0(t)\delta_{ij} - i \sum_{n=1}^{\infty} \frac{(-i\lambda)^n}{n!} \int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_n \\ \times \langle \Phi_0 | T[u_i(t)u_j(0)H_1(t_1) \ldots H_1(t_n)] | \Phi_0 \rangle. \end{aligned} \tag{3.1.24}$$