Physics 204A HW # 6 SOLUTIONS

Physics 204A, Fall 2010, Problem Set 6

[1.] [Problem 4, UCD Qualifying Exam 2000] Obtain the Fourier expansions of the following functions:

[15 pts]

$$\begin{aligned} f(x) &= e^{-x} \qquad 0 < x < 1 \\ f(x) &= 0 \qquad \text{otherwise} \end{aligned}$$

and

$$\begin{aligned} g(x) &= e^{-x} \qquad 0 < x < 1 \\ &= \text{extended periodically (with period 1) from } -\infty \to +\infty \end{aligned}$$

[2.] Let $F(\omega)$ be the Fourier transform of $f(x)$ and $G(\omega)$ be the Fourier transform of $g(x) = f(x+a)$. Show that

[10 pts]

$$G(\omega) = e^{+ia\omega}F(\omega)$$

[3.] Using the sequence

[10 pts]

$$\delta_n(x) = \frac{n}{\sqrt{\pi}}e^{-n^2x^2}$$

show that

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dk$$

Hint: Remember $\delta(x)$ is defined in terms of its behavior as part of an integrand.

[4.] The function $f(r)$ has a Fourier transform

[10 pts]

$$g(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\int f(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}d^3r = \frac{1}{(2\pi)^{3/2}k^2}$$

Determine $f(r)$. Hint: Use spherical polar coordinates in k-space.

[5.] Using partial fraction expansions, show that the inverse Laplace transform of $\frac{1}{(s+a)(s+b)}$ is $\frac{e^{-at}-e^{-bt}}{b-a}$ for $a \neq b$.

[10 pts]

[6.] A mass m is attached to one end of an unstretched spring, spring constant k. At time $t = 0$ the free end of the spring experiences a constant acceleration a away from the mass. Using Laplace transforms, find the position x of the mass m as a function of time and determine the limiting form of $x(t)$ for small t.

[15 pts]

[7.] Show that the Laplace transform of $\cosh(at)\cos(at)$ is $s^3/(s^4 + 4a^4)$.

[10 pts]

[8.] Solve Newton's equation for a mass m hit by an impulsive force,

[10 pts]

$$m\frac{d^2x}{dt^2} = P\delta(t)$$

using Laplace transforms.

[9.] A random walker moves on a discrete lattice in one dimension, starting at the origin, with equal probability of hopping one step to the left and right. Show that the mean square distance $\langle n^2 \rangle$ from the origin after N steps is proportional to N.

[10 pts]

P204A HW6 SOLUTIONS

[1] $f(x) = e^{-x} \quad 0 < x < 1$

(a) $f(x) = 0 \quad$ otherwise

Non periodic, Use fourier integral expansion given by inverse transform of $F(k)$, where $F(k)$ are coefficents of expansion

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-x} e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \frac{e^{-(ik+1)x}}{-(ik+1)}\Bigg|_0^1 = \frac{1}{\sqrt{2\pi}} \left[\frac{1 - e^{-(ik+1)}}{ik+1}\right]$$

So,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{+ikx}\, dk = \boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1 - e^{-(ik+1)}}{ik+1}\right] e^{+ikx}\, dk}$$

(b) Represent $g(x)$ (periodic) by a Fourier Series

$$g(x) = a_0 + \sum_{n=1} \left[a_n \cos(2\pi n x) + b_n \sin(2\pi n x)\right]$$

$$a_0 = \frac{1}{L}\int_0^L g(x)\, dx = \int_0^1 e^{-x}\, dx = -e^{-x}\Big|_0^1 = 1 - e^{-1}$$

$$a_n = \frac{2}{L}\int_0^L g(x) \cos(2\pi n x)\, dx = \int_0^1 e^{-x}\left[e^{i2\pi n x} + e^{-i2\pi n x}\right] dx$$

$$= \left[\frac{e^{x(-1+2\pi i n)}}{-1+2\pi i n} + \frac{e^{-x(1+2\pi i n)}}{-(1+2\pi i n)}\right] = \frac{e^{-1} - 1}{-1+2\pi i n} - \frac{e^{-1} - 1}{1+2\pi i n}$$

$$= \frac{(e^{-1} - 1)(1 + 2\pi i n + 1 - 2\pi i n)}{-(1 + 4\pi^2 n^2)} = \boxed{\frac{2(1 - e^{-1})}{1 + 4\pi^2 n^2}}$$

(b) $b_n = \frac{2}{L}\int_0^L g(x)\sin(2\pi n x)\,dx = \int_0^1 e^{-x}\frac{1}{i}\left[e^{i2\pi n x} - e^{-i2\pi n x}\right]$

$$= \frac{1}{i}\left[\frac{e^{x(-1+2\pi i n)}}{-1+2\pi i n} - \frac{e^{-x(1+2\pi i n)}}{-(1+2\pi i n)}\right]\Bigg|_0^1$$

$$= \frac{1}{i}\left[\frac{e^{-1}-1}{-1+2\pi i n} + \frac{e^{-1}-1}{1+2\pi i n}\right] = \frac{4\pi n(1-e^{-1})}{1+4\pi^2 n^2}$$

Putting it together,

$$\boxed{g(x) = (1-e^{-1}) + (1-e^{-1})\left[\sum_{n=1}^{\infty}\frac{2\cos(2\pi n x)}{1+4\pi^2 n^2} + \sum_{n=1}^{\infty}\frac{4\pi n\sin(2\pi n x)}{1+4\pi^2 n^2}\right]}$$

[2]

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx$$

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f(x+a)\, dx$$

Let $y = x+a$

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega(y-a)} f(y)\, dy$$

$$= e^{ia\omega} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega y} f(y)\, dy$$

$$\boxed{G(\omega) = e^{+ia\omega} F(\omega)}$$

[3] Assume that:

$$\delta(x) = \lim_{n\to\infty} \delta_n(x) = \lim_{n\to\infty} \frac{n}{\sqrt{\pi}}\, e^{-n^2x^2}$$

Take the Fourier transform of $\delta_n(x)$

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta_n(x)\, e^{+ikx}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{n}{\sqrt{\pi}}\, e^{-n^2x^2} e^{+ikx}\, dx$$

$$= \frac{1}{2\pi}\,\frac{n}{\pi^{1/2}} \int_{-\infty}^{\infty} e^{-n^2x^2+ikx}\, dx$$ Complete the square

$$= \frac{1}{2\pi}\,\frac{n}{\pi^{1/2}} \int_{-\infty}^{\infty} e^{-\left(nx-\frac{ik}{2n}\right)^2 - k^2/4n^2}\, dx$$

$$= \frac{1}{2\pi}\,\frac{n}{\pi^{1/2}}\, e^{-k^2/4n^2} \int_{-\infty}^{\infty} \left(\frac{1}{n}\right) e^{-y^2}\, dy$$

$\left(\text{where } y = nx - \frac{ik}{2n}\right)$

$$F(k) = \frac{1}{2\pi}\,\frac{\cancel{n}}{\pi^{1/2}}\left(\frac{1}{\cancel{n}}\right) e^{-k^2/4n} \left[\cancel{\pi^{1/2}}\right] = \frac{1}{2\pi}\, e^{-k^2/4n}$$

$F(k)\, e^{-ikx}$ [(FT of $F(k)$) $= \delta(x)$]

$$\delta(x) = \lim_{n\to\infty}\left[\delta_n(x)\right] = \lim_{n\to\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi}\, e^{-k^2/4n}\, e^{-ikx}\, dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{1}\, e^{-ikx}\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\, dk$$

So $$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\, dk$$

[4] $$f(r) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} g(\vec{k})\, e^{-i\vec{k}\cdot\vec{r}}\, d^3k$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{3/2}} \frac{1}{k^2}\, e^{-i\vec{k}\cdot\vec{r}}\, d^3k$$

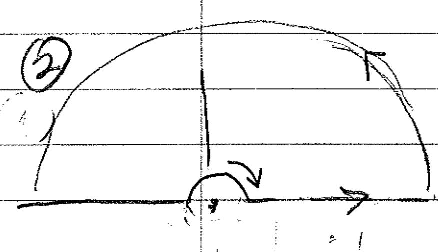
$$= \frac{1}{(2\pi)^3} \int \frac{1}{k^2}\, e^{-i\vec{k}\cdot\vec{r}}\, k^2 \sin\theta \, d\theta\, d\phi\, dk$$

$$= \frac{1}{(2\pi)^3}\left[\int_0^{2\pi} d\phi\right]\left[\int_0^{\infty}\int_0^{\pi} e^{-ikr\cos\theta} \sin\theta\, d\theta\, dk\right]$$

Orient r along z-axis $\Longrightarrow$ $i\vec{k}\cdot\vec{r} = ikr\cos\theta$ & assume $k, r > 0$

$$= \frac{2\pi}{(2\pi)^3}\int_0^{\infty} dk \left[\frac{e^{-ikr\cos\theta}}{ikr}\Big|_0^{\pi}\right]$$

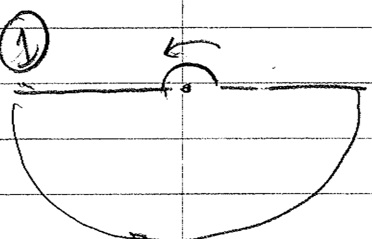
(2)



sin(kr)

$$= \frac{1}{(2\pi)^2}\int_0^{\infty} dk\, \frac{1}{ikr}\left[e^{+ikr} - e^{-ikr}\right]$$

(1)



$$= \frac{1}{4\pi^2 i r}\left[\int_0^{\infty}\left(\frac{e^{ikr}}{k} - \frac{e^{-ikr}}{k}\right) dk\right]$$

$$= \frac{1}{4\pi^2 i r}\left\{\frac{1}{2}\left[\underbrace{\int_{-\infty}^{\infty}\frac{e^{ikr}}{k}\, dk}_{(2)} - \underbrace{\int_{-\infty}^{\infty}\frac{e^{-ikr}}{k}\, dk}_{(1)}\right]\right\}$$

$$= \frac{1}{4\pi^2 i r}\left[\frac{1}{2}(\pi i)\, \underbrace{e^{ikr}}_{=1}\Big|_{k=0} - \frac{1}{2}(-\pi i)\, \underbrace{e^{-ikr}}_{=1}\Big|_{k=0}\right]$$

$$f(r) = \frac{1}{4\pi^2 i r}(\pi i) = \boxed{\frac{1}{4\pi r}}$$

(6)

[5] $\frac{1}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b}$

$\Rightarrow A(s+b) + B(s+a) = s(A+B) + Ab + Ba = 1$

$\Rightarrow A = -B \qquad Ab - Aa = 1 \Rightarrow \boxed{A = \frac{1}{b-a}}$

$\boxed{B = -\frac{1}{b-a}}$

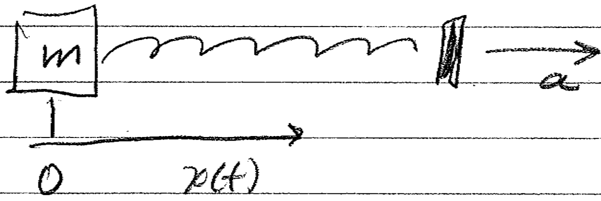
$F(s) = \left(\frac{1}{b-a}\right)\left(\frac{1}{s+a}\right) - \left(\frac{1}{b-a}\right)\left(\frac{1}{s+b}\right)$

$\Downarrow$ ~~Transform~~

$\underline{f(t)} = \frac{1}{b-a} e^{-at} - \frac{1}{b-a} e^{-bt} = \boxed{\frac{e^{-at} - e^{-bt}}{b-a}}$

$a \neq b$

[6]



distance = $\frac{1}{2} a t^2$
(constant a)

Let $k/m = \omega^2$ and let's assume mass is at rest at $x=0$ at time $t=0$ ($x(0)=0$, $\dot{x}(0)=0$)

Equation of Motion: $m\ddot{x}(t) = -k\left(x(t) - \frac{1}{2} a t^2\right)$

OR $\ddot{x}(t) + \omega^2 x(t) = \frac{1}{2} a t^2$

Laplace transform $\Downarrow$

$$-\cancel{\ddot{x}(0)} - s\cancel{\dot{x}(0)} + s^2 F(s) + \omega^2 F(s) = \frac{\omega^2 a}{2} \frac{\Gamma(3)}{s^3}$$

$$F(s) = \omega^2 a \left(\frac{1}{s^2+\omega^2}\right)\left(\frac{1}{s^3}\right)$$

Now, find partial fraction equivalent:

$$\frac{1}{(s^2+\omega^2)s^3} = \left(\frac{As+B}{s^2+\omega^2}\right) + \left(\frac{Cs^2+Ds+E}{s^3}\right)$$

$$\Rightarrow As^4 + Bs^3 + Cs^4 + Ds^3 + Es^2 + \omega^2 Cs^2 + \omega^2 Ds + \omega^2 E = 1$$

$$\Rightarrow \boxed{E = 1/\omega^2,\ A = -C,\ B = -D,\ E = -\omega^2 C,\ D = 0}$$

$$\Rightarrow C = -\frac{1}{\omega^2}E = -1/\omega^4,\ B = 0,\ A = 1/\omega^4$$

$$\Rightarrow \frac{1}{(s^2+\omega^2)s^3} = \frac{(1/\omega^4)s}{s^2+\omega^2} + \frac{(-1/\omega^4)s^2 + \frac{1}{\omega^2}}{s^3}$$

[6] cont'd

$$F(s) = \left(\frac{w^2 a}{w^4}\right)\left[\frac{s}{s^2+w^2} - \frac{1}{s} + \frac{w^2}{s^3}\right]$$

$$= \frac{a}{w^2}\left[\frac{s}{s^2+w^2} - \frac{1}{s} + \frac{w^2}{s^3}\right]$$

Inverse transforms: $\mathcal{L}^{-1}\left(\frac{s}{s^2+w^2}\right) = \cos(wt)$

$\mathcal{L}^{-1}(1/s^3) = \frac{t^2}{2}$ $\quad \mathcal{L}^{-1}(1/s) = 1$

$$\boxed{x(t) = \frac{a}{w^2}\cos(wt) - \frac{a}{w^2} + \frac{1}{2}at^2}$$

For small t, $\cos(wt) \approx \left(1 - \frac{(wt)^2}{2} + \frac{(wt)^4}{4!} - \dots\right)$

$$x(t) = \frac{a}{w^2}\left[1 - \frac{(wt)^2}{2} + \frac{(wt)^4}{4!}\right] - \frac{a}{w^2} + \frac{1}{2}at^2$$

$$x(t) = -\frac{1}{2}at^2 + \frac{a w^2 t^4}{4!} + \frac{1}{2}at^2$$

$$\approx \frac{a w^2 t^4}{4!}$$

so, we must go to 4th order for $x(t)$!

At small t, $x(t) \approx 0$

[7] $f(t) = \cosh(at)\cos(at) = \left(\frac{e^{at}+e^{-at}}{2}\right)\left(\frac{e^{iat}+e^{-iat}}{2}\right)$

$$F(s) = \int_0^\infty f(x)\, e^{-st}\, ds = \frac{1}{2}\int_0^\infty \left[e^{-(s-a)t}\cos at + e^{-(s+a)t}\cos at\right]$$

Now, $\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}$

AND:

$$\mathcal{L}(e^{-at}\cos at) = \frac{s+a}{(s+a)^2+a^2}$$

$$\mathcal{L}(e^{at}\cos at) = \frac{s-a}{(s-a)^2+a^2}$$

$$F(s) = \frac{1}{2}\left[\mathcal{L}(e^{-at}\cos at) + \mathcal{L}(e^{at}\cos at)\right]$$

$$= \frac{1}{2}\left[\frac{s+a}{(s+a)^2+a^2} + \frac{s-a}{(s-a)^2+a^2}\right]$$

$(s-a)(s^2-a^2) + 2sa^2 \qquad (s+a)(s^2-a^2)$

$$= \frac{1}{2}\left[\frac{(s+a)(s-a)^2 + sa^2 + \cancel{a^3} + (s+a)^2(s-a) + sa^2 - \cancel{a^3}}{(s+a)^2(s-a)^2 + a^2\left[(s+a)^2+(s-a)^2\right] + a^4}\right]$$

$$= \frac{1}{2}\left[\frac{s^3 - as^2 - a^2 s + a^3 + 2sa^2 + s^3 + as^2 - a^2 s - a^3}{(s^2-a^2)^2 + a^2\left[2s^2+2a^2\right] + a^4}\right]$$

$$= \frac{1}{2}\left[\frac{2s^3}{s^4 - \cancel{2a^2s^2} + a^4 + \cancel{2a^2s^2} + 2a^4 + a^4}\right]$$

$$\boxed{F(s) = \left[\frac{s^3}{s^4+4a^4}\right]}$$

[8] $m \dfrac{d^2x}{dt^2} = P\,\delta(t-t_1)$ Impulse at time $= t_1$ where $t_1 > 0$

$$\ddot{x}(t) = \frac{P}{m}\,\delta(t-t_1)$$

$$-\dot{x}(0) - s\,x(0) + s^2 F(s) = \frac{P}{m}\int_0^\infty e^{-st'}\,\delta(t'-t_1)\,dt'$$

$$s^2 F(s) = \frac{P}{m}\,e^{-st_1}$$

$$F(s) = \frac{P}{m}\,\frac{\boxed{e^{-st_1}}}{s^2}$$ $\longrightarrow$ shifted function

step function

$$\boxed{x(t) = \frac{P}{m}\,t\,\theta(t-t_1)}$$ where $\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$

So mass travels at constant velocity (P/m) after $t = t_1$.

Actual problem was

$$m\,\frac{d^2x}{dt^2} = P\,\delta(t)$$

$$\Rightarrow s^2 F(s) = \frac{P}{m} \quad \Rightarrow F(s) = \frac{P}{m}\left(\frac{1}{s^2}\right)$$

$$\mathcal{L}^{-1}(F(s)) = \boxed{x(t) = \frac{P}{m}\,t}$$

[9] There are several ways to do this problem.

(1) Brute force by powering through binomial expansion

$$\langle n^2 \rangle = \sum \frac{N!}{n_e! \, n_r!} p^{n_e} p^{n_r} (n_e - n_r)^2$$

where n_e = # of left moves
n_r = # of right moves

SIMPLER:

(2) Note that moves are <u>independent</u> random variables!
$S_i = +1$ (right move) or -1 (left move) where
$\langle S_i \rangle = 0$, $\langle S_i^2 \rangle = 1$, and $\langle S_i S_j \rangle = \langle S_i \rangle \langle S_j \rangle = 0$ (for $i \neq j$)

So $n = \sum_i^N S_i$ (sum of left & right moves)

$$\langle n_N^2 \rangle = \langle (\sum_i^N S_i)(\sum_j S_i) \rangle = \langle \sum_{ij}^N S_i S_j \rangle = \sum_i^N \langle S_i^2 \rangle = \underline{\underline{N}}$$

(3) Let d_n = distance from origin after n moves

Then $d_n = \begin{cases} d_{n-1} + 1 \\ d_{n-1} - 1 \end{cases}$ Each with probability $(1/2)$

$$\Rightarrow \langle d_n^2 \rangle = \tfrac{1}{2}\langle (d_{n-1}+1)^2 \rangle + \tfrac{1}{2}\langle (d_{n-1}-1)^2 \rangle$$

$$\Rightarrow \langle d_n^2 \rangle = \langle d_{n-1}^2 \rangle + 1$$

If $\langle d_1^2 \rangle = 1$, then by recursion

$$\langle d_n^2 \rangle = N$$