Physics 204A, Fall 2010, Problem Set 5

[1.] Consider a damped harmonic oscillator subject to a “half-wave” driving force,

\[ F(t) = \sin \omega t \quad 0 < t < \pi/\omega \]
\[ F(t) = 0 \quad \pi/\omega < t < 2\pi/\omega . \]

where \( F(t) \) is periodic in \( t \) with period \( T = 2\pi/\omega \). Write down the long time form for \( x(t) \). Just as in the sawtooth wave problem [2b] of assignment 4, plot the 2-, 4-, 6-, 8- term representations of the drive force \( F(t) \). Compare how well they do with the sawtooth case. Why is there a marked difference in convergence?

[2.] In class we derived a condition for a second order differential operator,

\[ \mathcal{L} = p_0(t) \frac{d^2}{dt^2} + p_1(t) \frac{d}{dt} + p_2(t) \]

to be Hermitian. As one example, \( \mathcal{L} = d^2/dt^2 \) is Hermitian. Is the differential operator

\[ \mathcal{L} = m \frac{d^2}{dt^2} + \gamma \frac{d}{dt} + m\omega_0^2 \]

encountered in the damped oscillator problem Hermitian? What are its eigenfunctions? Are they complete?

[3.] Prove

\[ \frac{1}{L} \sum_{l=1}^{L} e^{2\pi i (n-m) t/L} = \delta_{nm} \]

[4.] Show that the general solution \( x(t) \) of a damped driven oscillator subject to an even force \( F(t) \) (i.e. only cosines in the Fourier expansion) of period \( T \) is given by,

\[ x(t) = \int_0^T dt' G(t, t') F(t') \]
\[ G(t, t') = \sum_n \frac{\cos \omega_n t' \cos (\omega_n t - \delta_n)}{\left( m^2(\omega_0^2 - \omega_n^2)^2 + \gamma^2 \omega_n^2 \right)^{1/2}} \]
\[ \delta_n = \tan^{-1}(\gamma \omega_n/m(\omega_0^2 - \omega_n^2)) \]

with \( \omega_n = 2\pi n/T \). You may use all the equations we derived in class. The function \( G(t, t') \) is called the Green’s function. This result is quite remarkable in the sense that it provides an expression for \( x(t) \) which is valid for any \( F(t) \). (This generality is the whole point of computing a Green’s function.)

[5.] Similarly, prove that the general solution \( \Psi(x, t) \) to a quantum problem with any initial wavefunction \( \Psi(x, 0) \) is,

\[ \Psi(x, t) = \int dx' G(x, x', t) \Psi(x', 0) \]
\[ G(x, x', t) = \sum_n \phi_n^*(x') \phi_n(x) e^{-iE_nt/n} \]

where \( H\phi_n(x) = E_n\phi_n(x) \) are the eigenfunctions and eigenvalues of the time independent Schroedinger equation.
Damped harmonic oscillator with "half-wave" sine driving force

\[ m \ddot{x}(t) + m \omega_0^2 x(t) + \gamma \dot{x}(t) = F(t) \]

where \( F(t) = \begin{cases} \sin \omega t & 0 < t < \frac{\omega}{2} \\ 0 & \frac{\omega}{2} < t < \frac{2\omega}{\pi} \end{cases} \)

**Fourier Series for \( F(t) \):**

\[ F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n \omega t) + \sum_{n=1}^{\infty} b_n \sin(n \omega t) \]

\[ a_0 = \frac{w}{\pi} \int_{0}^{\pi/\omega} \sin \omega t \, dt = \frac{w}{\pi} \left[ -\frac{1}{w} \cos \omega t \right]_0^{\pi/\omega} = \frac{2}{\pi} \]

\[ a_n = \frac{w}{\pi} \int_{0}^{\pi/\omega} \sin \omega t \cos(n \omega t) \, dt \]

\[ \text{Use } \sin \alpha \cos \beta = \frac{1}{2} \left[ \sin(\alpha + \beta) + \sin(\alpha - \beta) \right] \]

\[ = \frac{w}{2\pi} \int_{0}^{\pi/\omega} \frac{1}{2} \left[ \sin((n+1)\omega t) + \sin((1-n)\omega t) \right] \, dt \]

\[ = \frac{w}{2\pi} \left[ -\frac{1}{w(n+1)\omega} \cos((n+1)\omega t) \right]_0^{\pi/\omega} - \frac{1}{w(1-n)\omega} \cos((1-n)\omega t) \right]_0^{\pi/\omega} \]

\[ = \frac{1}{\pi} \left( \frac{2}{n+1} + \frac{2}{1-n} \right) \text{ for } n \text{ even} \]

\[ \text{for } n \text{ odd} \]

So

\[ a_n = \frac{1}{\pi} \left[ \frac{2}{1-n^2} \right] \text{ for } n \text{ even} \]
\[
\begin{align*}
\hat{b}_n &= \frac{w}{\pi} \int_0^{\pi/w} \sin(nwt) \sin(nwt) \, dt \\
&= \frac{w}{2\pi} \int_0^{\pi/w} \left[ \cos((1-n)wt) - \cos((n+1)wt) \right] \, dt \\
&= \frac{w}{2\pi} \left[ \frac{1}{(1-n)w} \sin((1-n)wt) \right]_0^{\pi/w} - \frac{1}{(n+1)w} \sin((n+1)wt) \right]_0^{\pi/w} \\
&= \frac{w}{2\pi} \frac{1}{(1-n)w} \sin((1-n)(\pi/w)) - \frac{1}{(n+1)w} \sin((n+1)(\pi/w)) \\
&= \frac{1}{2}\left[ \frac{1}{n} \sin(n\pi) \right] \\
&= \frac{1}{2} \quad \text{for even } n \\
&= 0 \quad \text{for odd } n \\
\end{align*}
\]

Finally,
\[
F(t) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(nwt)}{(1-n^2)} + \frac{1}{2} \sin wt
\]

Plots for 2-, 4-, 6-, 8- term representation of \( F(t) \) are attached.

Fourier sine for half-wave sawtooth converges much faster than sawtooth wave, because \( F(t) \) is continuous and sawtooth wave is discontinuous.
FS for sine half-wave: $F(t) = \sin(wt), \ 0 < t < \pi/w$
import numpy as np
from math import pi, cos, sin, sqrt
from matplotlib.pyplot import *

# Build array of 'x' values to compute
x = linspace(0, 2, 101) * pi

# Function to compute fourier series for
# variable number of terms (num_terms)
# x is an array of input values

def FS_tot(num_terms, x):
    w = 1.0
    total = 0.0
    for n in range(num_terms+1):
        if n == 0:
            total = 1./pi
        continue
        if n == 1:
            total += .5 * sin(w*x)
        else:
            m = 2.*n - 2,
            # Compute only even cos terms
            total += -2. / pi) * (1./((m**2 - 1)) * cos(m*w*x)
    return total

# Allow FS_tot function to work on entire array of values
FS_tot_vect = vectorize(FS_tot)

# Build FS arrays for 2, 4, 6, and 8 term series
f_exact = sin(x)
f_exact[50:] = 0.0

f_2 = FS_tot_vect(2, x)
f_4 = FS_tot_vect(4, x)
f_6 = FS_tot_vect(6, x)
f_8 = FS_tot_vect(8, x)

# Plot FS on same diagram
plot(x, f_exact, 'b-', x, f_2, 'm:', x, f_4, 'r-', x, f_6, 'g--', x, f_8, 'r:');
title('FS for sine half-wave: F(t)=sin(w*t), 0 < t < pi/w')
legend(['f_exact', 'f2', 'f4', 'f6', 'f8'], 'loc = upper right')
xlabel('t')
ylabel('F(t)')
grid()
savefig('FS_HW5_1.pdf')
clos
Long form of \( x(t) \) is given by (class notes)

\[
x(t) = \sum_{n=0}^{\infty} \frac{\alpha_n \cos \left( \omega_n t - \delta_n \right)}{(m^2(\omega_0^2 - \omega_n^2)^2 + \gamma_n^2 \omega_n^2)^{1/2}} + \sum_{n=0}^{\infty} b_n S_n
\]

where \( \omega_n = \frac{2\pi n}{T} = \sqrt{\omega_n} \)

and \( \delta_n = \tan^{-1} \left( \frac{\gamma_n \omega_n}{m(\omega_0^2 - \omega_n^2)} \right) \)

\[x(t) = \frac{1}{\pi \sqrt{m \omega_0^2}} + \frac{1}{2 \left[ m^2(\omega_0^2 - \omega_n^2)^2 + \gamma_n^2 \omega_n^2 \right]^{1/2}} \]

\[+ \frac{2}{m} \sum_{n=1}^{\infty} \frac{\cos (4\pi n t)}{(1 - 4n^2)^2 \left[ m^2(\omega_0^2 - \omega_n^2)^2 + \gamma_n^2 \omega_n^2 \right]^{1/2}} \]
$$L = m \frac{d^2}{dt^2} + \gamma \frac{d}{dt} + mw_0^2$$

Is it Hermitian?  
$$p_0 = m \quad p_1 = \gamma$$

$$p_1 \neq p_0' = 0 \quad \text{Not hermitian for} \quad \gamma \neq 0$$

Eigenvalue equation is:

$$L u(t) = \lambda u(t)$$

or

$$m \frac{d^2 u(t)}{dt^2} + \gamma \frac{d u(t)}{dt} + (mw_0^2 - \lambda) u(t) = 0$$

Let's try: 
$$u(t) = Ae^{\lambda t}$$

$$\Rightarrow \quad m \lambda^2 Ae^{\lambda t} + \gamma \lambda Ae^{\lambda t} + (mw_0^2 - \lambda) Ae^{\lambda t} = 0$$

$$\Rightarrow \quad m \lambda^2 + \gamma \lambda + (mw_0^2 - \lambda) = 0$$

Solving for $\lambda$:

$$\lambda_{1,2} = -\frac{\gamma \pm \sqrt{\gamma^2 - 4mw(mw_0^2 - \lambda)}}{2m}$$

$$\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$$ are eigenfunctions for each real $\lambda$

Hard to prove, but I suspect a continuous (2) of these functions would "cover" the space of real, continuous functions.
Prove \( \frac{1}{L} \sum_{l=1}^{l=L} e^{2\pi i (n-m) l/L} = \delta_{mn} \)

Assume \( n=m \):
\[
\sum_{l=1}^{L} e^{2\pi i l} = \frac{L}{L} = 1 \quad \checkmark
\]

Assume \( n \neq m \):
\[
\frac{1}{L} \left[ \frac{1 - e^{2\pi i (n-m)}}{1 - e^{2\pi i (n-m)/L}} \right] = \frac{1}{L} \left[ \frac{1 - (1)^{n-m}}{1 - (1)^{(n-m)/L}} \right]
\]

\[
\text{(where we use geometric sum formula: } \sum_{l=1}^{L} a^l = \frac{a^{l+1} - a}{1 - a} \text{)}
\]
\[
= 0 \quad \checkmark
\]

So \( \Box \) = \( \delta_{mn} \)
Note: This solution is from first principles.

A simpler (less rigorous?) approach is to use

\[ \sum_{n} a_n \int F(t) \cos(\omega nt) \, dt \]

and

\[ \sum_{m=1}^{\infty} a_m \cos(\omega mt) \]

Assume \( F(t) = \sum_{n} a_n \cos(\omega nt) \)

where \( \omega_n = \frac{2\pi n}{T} \)

\[ G(t, t') = \sum_{n} \frac{\cos(\omega nt') \cos(\omega nt - \delta_n)}{(m^2(\omega_0^2 - \omega_n^2)^2 + \gamma^2 \omega_n^2)^{1/2}} = \frac{D}{\text{denominator}} \]

where \( \delta_n = \tan^{-1} \left( \frac{\gamma \omega_n}{m(\omega_0^2 - \omega_n^2)} \right) \)

For damped driven oscillators:

\[ \ddot{x}(t) + \gamma \dot{x}(t) + m\omega_0^2 x = F(t) \]

Show that \( x(t) = \frac{1}{T} \int_{0}^{T} dt' G(t, t') F(t') \)

is the general solution.

First,

\[ L \, G(t, t') = \sum_{n} \frac{\cos(\omega nt')}{D} \left[ -m\omega_n^2 \cos(\omega nt - \delta_n) + \gamma \omega_n \sin(\omega nt - \delta_n) \right. \]

\[ \left. + m\omega_0^2 \cos(\omega nt - \delta_n) \right] \]

Now,

\[ J \, x(t) = \frac{2}{T} \int_{0}^{T} dt' G(t, t') F(t') \]

operates only on \( t \).

\[ = \frac{2}{T} \int_{0}^{T} dt' \sum_{n} a_n \cos(\omega nt') \sum_{m} a_m \cos(\omega mt') \]

(next page)
\[ L = \frac{2}{\pi} \sum_{n,m} \cos(\omega t') \cos(\omega t) \frac{a_n}{D} \]
[4] continued

Finally, we get:

$$D^2 x(t) = \sum_{n} \frac{a_n}{D^2} \left\{ \left[ m(w_0^2 - w_n^2) + 2 \delta w_n^2 \right] \cos wt \right. $$

$$+ \left. \left[ m(w_0^2 - w_n^2) \delta w_n - m(w_0^2 - w_n^2) \delta w_n \sin wt \right] \right\}$$

$$Fx(t) = \sum_{n} a_n \cos wt = F(t)$$

So \( x(t) \) is a general solution to the differential equation.
Let $\phi_n(x)$ be eigenfunctions, values

$$H \phi_n(x) = E_n \phi_n(x)$$

Show that $\psi(x,t) = \int dx' G(x,x',t) \psi(x',0)$

where $G(x,x',t) = \sum_n \phi_n^*(x') \phi_n(x) e^{-i E_n t / \hbar}$

is a general solution for any initial $\psi(x,0)$

We must show that $\psi(x,t)$ satisfies the TDSE

$$\frac{i}{\hbar} \frac{\partial}{\partial t} \langle \psi \rangle = H \langle \psi \rangle$$

First, we can expand $\psi(x,0)$ in the $\phi_n(x)$ eigenfunctions:

$$\psi(x,0) = \sum_n a_n \phi_n(x)$$

Also, let $\phi_n(x,t) = e^{-i E_n t / \hbar} \phi_n(x)$ (time-evolved)

$$\frac{\partial \psi}{\partial t} = \frac{i}{\hbar} \frac{\partial}{\partial t} \int dx' G(x,x',t) \psi(x',0)$$

$$= \frac{i}{\hbar} \frac{\partial}{\partial t} \left( \sum_n \phi_n^*(x') \phi_n(x) e^{-i E_n t / \hbar} \right) \sum_m a_m \phi_m(x)$$

$$= \int dx' \sum_n \sum_m \left[ \left( -\frac{i E_n}{\hbar} \right) e^{-i E_n t / \hbar} \phi_n(x') \phi_n(x) a_m \phi_m(x') \right]$$

$$= \sum_n \sum_m \left( -\frac{i E_n}{\hbar} \right) e^{-i E_n t / \hbar} a_m \phi_n(x) \phi_m(x')$$

$$= \sum_n \left( -\frac{i E_n}{\hbar} \right) e^{-i E_n t / \hbar} \phi_n(x)$$

So,

$$\frac{i}{\hbar} \frac{\partial \psi}{\partial t} = \sum_n a_n E_n \phi_n(x,t)$$
Now, \( H \Psi(x, t) \)

\[
H \Psi(x, t) = \int dx' G(x, x', t) \Psi(x', 0)
\]

\( \rightarrow \) operates on \( x \) only!

\[
= \int dx' \frac{\hbar}{i} \left( \sum_n \phi_n^*(x') \phi_n(x) e^{-iE_n t/\hbar} \right) \left( \sum_m a_m \phi_m(x') \right)
\]

\[
= \int dx' \sum_{nm} \phi_n(x') \phi_m^*(x') \left( \frac{\hbar}{i} \right) \frac{e^{-iE_n t/\hbar}}{E_n} \delta_{nm}
\]

\[
= \sum_n \delta_{nm} \frac{\hbar}{i} \frac{e^{-iE_n t/\hbar}}{E_n} \phi_n(x)
\]

\[
\left| H \Psi(x, t) \right| = \sum_n a_n E_n \phi_n(x, t)
\]

So \( i\hbar \frac{\partial \Psi}{\partial t} = H \Psi \) and \( |\Psi\rangle \) is a general solution for any initial \( \Psi(x, 0) \).