P204A HW #5 SOLUTIONS

FALL, 2010

Physics 204A, Fall 2010, Problem Set 5

[30 pts]

[1.] Consider a damped harmonic oscillator subject to a "half-wave" driving force,

$$\begin{aligned} F(t) &= \sin \omega t \qquad & 0 < t < \pi/\omega \\ F(t) &= 0 \qquad & \pi/\omega < t < 2\pi/\omega \,. \end{aligned}$$

where $F(t)$ is periodic in t with period $T = 2\pi/\omega$. Write down the long time form for $x(t)$. Just as in the sawtooth wave problem [2b] of assignment 4, plot the $2-, 4-, 6-, 8-$ term representations of the drive force $F(t)$. Compare how well they do with the sawtooth case. Why is there a marked difference in convergence?

[20 pts]

[2.] In class we derived a condition for a second order differential operator,

$$\mathcal{L} = p_0(t)\ d^2/dt^2 + p_1(t)\ d/dt + p_2(t)$$

to be Hermitian. As one example, $\mathcal{L} = d^2/dt^2$ is Hermitian. Is the differential operator

$$\mathcal{L} = m\ d^2/dt^2 + \gamma\ d/dt + m\omega_0^2$$

encountered in the damped oscillator problem Hermitian? What are its eigenfunctions? Are they complete?

[10 pts]

[3.] Prove

$$\frac{1}{L}\sum_{l=1}^{L} e^{2\pi i\,(n-m)\,l\,/\,L} = \delta_{nm}$$

[20 pts]

[4.] Show that the general solution $x(t)$ of a damped driven oscillator subject to an even force $F(t)$ (*i.e.* only cosines in the Fourier expansion) of period T is given by,

$$\begin{aligned} x(t) &= \int_0^T dt'\ \mathcal{G}(t,t')\,F(t') \\ \mathcal{G}(t,t') &= \sum_n \frac{\cos\omega_n t'\ \cos\left(\omega_n t - \delta_n\right)}{\left(\,m^2(\omega_0^2-\omega_n^2)^2 + \gamma^2\omega_n^2\,\right)^{1/2}} \\ \delta_n &= \tan^{-1}(\,\gamma\omega_n/m(\omega_0^2-\omega_n^2)\,) \end{aligned} \tag{1}$$

with $\omega_n = 2\pi n/T$. You may use all the equations we derived in class. The function $\mathcal{G}(t,t')$ is called the Green's function. This result is quite remarkable in the sense that it provides an expression for $x(t)$ which is valid for *any* $F(t)$. (This generality is the whole point of computing a Green's function.)

[20 pts]

[5.] Similarly, prove that the general solution $\Psi(x,t)$ to a quantum problem with *any* initial wavefunction $\Psi(x,0)$ is,

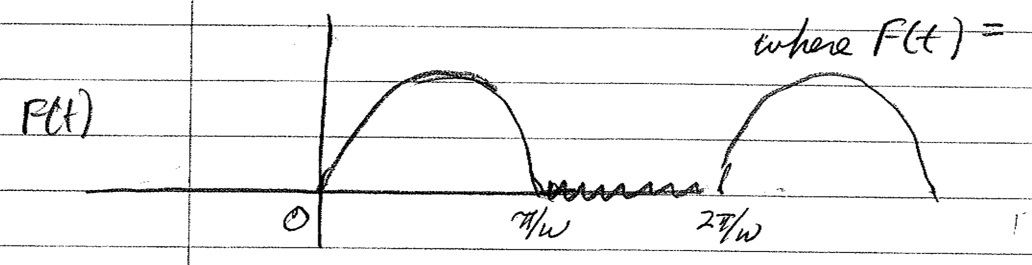
$$\begin{aligned} \Psi(x,t) &= \int dx'\,\mathcal{G}(x,x',t)\,\Psi(x',0) \\ \mathcal{G}(x,x',t) &= \sum_n \phi_n^*(x')\,\phi_n(x)\,e^{-iE_nt/\hbar} \end{aligned}$$

where $H\phi_n(x) = E_n\phi_n(x)$ are the eigenfunctions and eigenvalues of the time independent Schroedinger equation.

[1] Damped harmonic oscillator with "half-wave" sine driving force

$$m\ddot{x}(t) + m\omega_0^2 x(t) + \gamma\dot{x}(t) = F(t)$$

where $F(t) = \begin{cases} \sin\omega t & 0 < t < \pi/\omega \\ 0 & \pi/\omega < t < 2\pi/\omega \end{cases}$



Fourier Series for $F(t)$:

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$a_0 = \frac{\omega}{\pi}\int_0^{\pi/\omega} \sin\omega t \, dt = \frac{\omega}{\pi}\left[-\frac{1}{\omega}\cos\omega t\right]_0^{\pi/\omega} = \frac{2}{\pi}$$

$$a_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} \sin\omega t \cos(n\omega t)\, dt$$

[Use $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$]

$$= \frac{\omega}{\pi}\int_0^{\pi/\omega} \frac{1}{2}\left[\sin(n+1)\omega t + \sin(1-n)\omega t\right] dt$$

$$= \frac{\omega}{2\pi}\left[-\frac{1}{\omega(n+1)}\cos(n+1)\omega t\Big|_0^{\pi/\omega} - \frac{1}{\omega(1-n)}\cos(1-n)\omega t\Big|_0^{\pi/\omega}\right.$$

$$= \frac{1}{2\pi}\left\{\begin{matrix} \frac{2}{(n+1)} + \frac{2}{(1-n)} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{matrix}\right\}$$

So $\boxed{a_n = \frac{1}{\pi}\left[\frac{2}{1-n^2}\right] \quad \text{for } n \text{ even}}$

[1] continued:

Use identity:

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$b_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} \sin\omega t \sin n\omega t$$

$$= \frac{\omega}{2\pi}\int_0^{\pi/\omega} [\cos(1-n)\omega t - \cos(n+1)\omega t]\, dt$$

$$= \frac{\omega}{2\pi}\left[\frac{1}{(1-n)\omega}\sin(1-n)\omega t\Big|_0^{\pi/\omega} - \frac{1}{(n+1)\omega}\sin(n+1)\omega t\Big|_0^{\pi/\omega}\right]$$

But $\sin n\pi$ vanishes for all integers $n \Rightarrow$

So $\boxed{b_n = 0}$, except for term $n = 1$

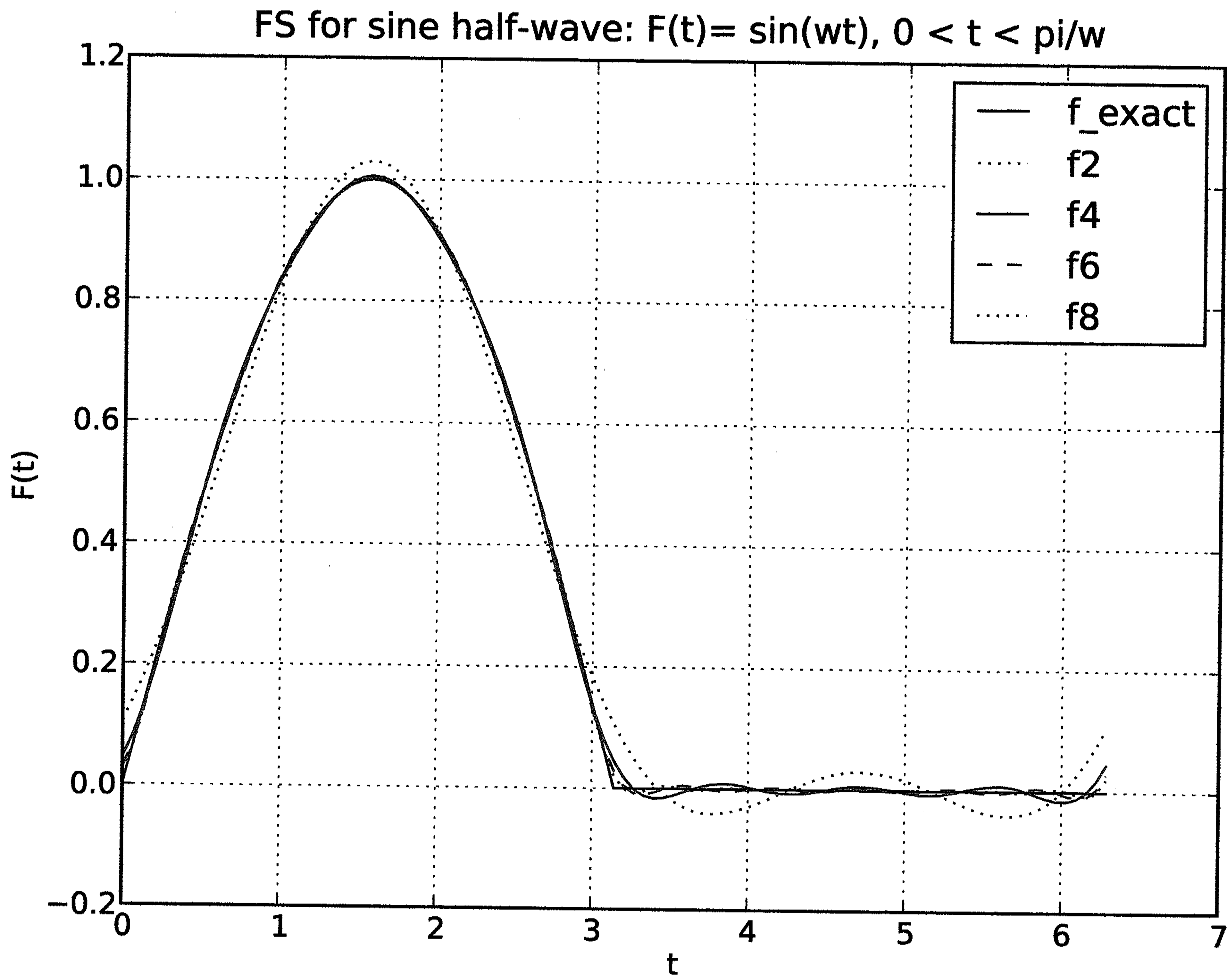
$$b_1 = \frac{\omega}{2\pi}\int_0^{\pi/\omega} \cos 0 \, dt = \frac{\omega}{2\pi}\left[\pi/\omega\right] = \boxed{\frac{1}{2}}$$

Finally,

$$\boxed{F(t) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n\ \mathrm{even}} \frac{\cos(n\omega t)}{(1-n^2)} + \frac{1}{2}\sin\omega t}$$

Plots for 2-, 4-, 6-, 8- term representation of $F(t)$ are attached.

Fourier series for half-wave sine converges much faster than sawtooth wave, because $F(t)$ is continuous and sawtooth wave is discontinuous.


FS for sine half-wave: F(t)= sin(wt), 0 < t < pi/w
F(t)
t
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0
1
2
3
4
5
6
7
f_exact
f2
f4
f6
f8

5


```
#!/usr/bin/env python
#  Physics 204A HW#5 Problem 1
#  For f(t) = sin(wt) 0< t < pi/w calculate FS for
#  x/pi = -1, -.98, -.96, ...,0, ..,1.0
#  Use 2,4,6,8 terms
import numpy as np
from math import pi, cos, sin, sqrt
from matplotlib.pylab import *
#
#   Build array of 'x' values to compute
#
x = linspace(0,2,101) * pi
#
#   Function to compute fourier series for
#   variable number of terms (num_terms)
#      x is an array of input values
#
def FS_tot(num_terms, x):
    w = 1.0
    total = 0.0
    for n in range(num_terms+1):
        if n == 0:
            total = 1./pi
            continue
        if n == 1:
            total += .5 * sin(w*x)
        else:
            m = 2.*n -2.     #  Compute only even cos terms
            total += -(2./pi) * (1./(m*m - 1))* cos(m*w*x)
    return total
#
#  Allow FS_tot function to work on entire array of values
#
FS_tot_vect = vectorize(FS_tot)
#
#  Build FS arrays for 2, 4, 6, and 8 term series
#
#  Fill up half wave values
f_exact = sin(x)
f_exact[ 50:] = 0.0

f_2 = FS_tot_vect(2,x)
f_4 = FS_tot_vect(4,x)
f_6 = FS_tot_vect(6,x)
f_8 = FS_tot_vect(8,x)
#
#   Plot FS on same diagram
#
plot(x,f_exact, 'b-', x, f_2, 'm:', x, f_4, 'r-', x, f_6, 'g--', x, f_8,'r:')
title('FS for sine half-wave: F(t)= sin(wt), 0 < t < pi/w')
l = legend(('f_exact','f2','f4','f6','f8'), loc ='upper right')
xlabel(' t ')
ylabel('F(t)')
grid()
savefig('FS_HW5_1.pdf')
clf()
```

[1] continued

Long form of $x(t)$ is given by (class notes)

$$x(t) = \sum_{n=0} \frac{a_n \cos(\omega_n t - \delta_n)}{(m^2(\omega_0^2 - \omega_n^2)^2 + \gamma_n^2 \omega_n^2)^{1/2}} + \sum_{n=0}^{\infty} b_n[\quad]$$

where $\omega_n = \frac{2\pi n}{T} = \boxed{\omega n}$

& $\delta_n = \tan^{-1}\left(\frac{\gamma \omega n}{m(\omega_0^2 - \omega^2 n^2)}\right)$

So

$$x(t) = \frac{1}{\pi m \omega_0^2} + \frac{1}{2} \frac{\sin(\omega t - \delta_1)}{[m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]^{1/2}}$$

$$+ \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n\omega t)}{(1-4n^2)} \left[\frac{1}{[m^2(\omega_0^2 - \omega^2 n^2)^2 + \gamma_n^2 \omega^2 n^2]}\right]^{1/2}$$

[2] $\mathcal{L} = m\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + m\omega_0^2$

Is it Hermetian? $p_0 = m$ $p_1 = \gamma$

$p_1 \neq p_0' = 0$ Not hermetian for $\gamma \neq 0$

Eigenvalue Equation is:

$$\mathcal{L}\, u(t) = \lambda\, u(t)$$

or $$m\frac{d^2u(t)}{dt^2} + \gamma\frac{d}{dt}u(t) + (m\omega_0^2 - \lambda)\, u(t) = 0$$

Let's try $u(t) = Ae^{\alpha t}$

$$\Rightarrow m\alpha^2 Ae^{\alpha t} + \gamma\alpha Ae^{\alpha t} + (m\omega_0^2 - \lambda)Ae^{\alpha t} = 0$$

$$\Rightarrow m\alpha^2 + \gamma\alpha + (m\omega_0^2 - \lambda) = 0$$

Solving for α: $$\alpha_1, \alpha_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4m(m\omega_0^2 - \lambda)}}{2m}$$

$\left\{ e^{\alpha_1(\lambda)t}, e^{\alpha_2(\lambda)t} \right\}$ are eigenfunctions for each real λ

Hard to prove, but I suspect a continuum (λ) of these functions would "cover" the space of real, continuous functions.

[3]

Prove $① = \frac{1}{L} \sum_{l=1}^{L} e^{2\pi i (n-m) l / L} = \delta_{mn}$

Assume $n = m$:

$① = \frac{1}{L} \sum_{l=1}^{L} e^{0} = \frac{L}{L} = 1$ ✓

Assume $n \neq m$

$① = \frac{e^{2\pi i (n-m)/L}}{L} \left[\frac{1 - e^{2\pi i (n-m)}}{1 - e^{2\pi i (n-m)/L}} \right] = \frac{1}{L} \left[\frac{1 - (1)^{n-m}}{1 - e^{2\pi i (n-m)/L}} \right]$

$\left(\text{where we use geometric sum formula} \quad \sum_{l=1}^{L} a^{l} = \frac{a^{1} - a^{L+1}}{1 - a} \right)$

$= 0$ ✓

So $① = \delta_{mn}$

<u>Note</u>: This solution is from "first principles". Simpler (less rigorous?) approach is to use $\left[a_n = \int F(t)\cos(\omega_n t)\right]$ and substitute!!

[4] Assume $F(t) = \sum_{m=1}^{\infty} a_m \cos(\omega_m t)$

where $\omega_n = 2\pi n/T$

$$G(t,t') = \sum_n \frac{\cos(\omega_n t')\cos(\omega_n t - \delta_n)}{\left(m^2(\omega_0^2-\omega_n^2)^2 + \gamma^2\omega_n^2\right)^{1/2}} = D \text{ denominator}$$

where $\delta_n = \tan^{-1}\left(\gamma\omega_n / m(\omega_0^2-\omega_n^2)\right)$

For damped driven oscillator:

$$\mathcal{L}\,x(t) = m\ddot{x} + \gamma\dot{x} + m\omega_0^2 x = F(t)$$

Show that $x(t) = \frac{2}{T}\int_0^T dt'\, G(t,t')\,F(t')$ is the general solution.

First,

$$\underset{\text{w/n } t}{\mathcal{L}}\, G(t,t') = \sum_n \frac{\cos(\omega_n t')}{D}\left[-m\omega_n^2\cos(\omega_n t-\delta_n) - \gamma\omega_n\sin(\omega_n t-\delta_n) + m\omega_0^2\cos(\omega_n t-\delta_n)\right]$$

$$= \sum_n \frac{\cos(\omega_n t')}{D}\left[m(\omega_0^2-\omega_n^2)\cos(\omega_n t-\delta_n) - \gamma\omega_n\sin(\omega_n t-\delta_n)\right]$$

Now, $\mathcal{L}\,x(t) = \mathcal{L}\,\frac{2}{T}\int_0^T dt'\, G(t,t')\,F(t')$ operates only on t!

$$= \frac{2}{T}\int_0^T dt'\, \mathcal{L}\,G(t,t')\left(\sum_m a_m\cos(\omega_m t')\right)$$

(next page)

[4] continued

$$= \frac{2}{T}\int_0^T dt' \sum_{n,m} \frac{\cos(\omega_n t')\cos(\omega_m t')\, a_m}{D}\Big[\qquad\qquad\Big]$$

But $\{\cos(\omega_n t)\}$ are orthogonal, so if we move integral over those terms we get

$$\int_0^T dt' \cos(\omega_n t')\cos(\omega_m t')dt' = \delta_{nm}\left(\frac{T}{2}\right)$$

$$\mathcal{L}x(t) = \left(\frac{2}{T}\right)\left(\frac{T}{2}\right)\sum_n \frac{a_n}{D}\left[m(\omega_0^2-\omega_n^2)\cos(\omega_n t-\delta_n) - \gamma\omega_n \sin(\omega_n t-\delta_n)\right] = A$$

Now, use
$$\begin{cases}\cos(\alpha-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ \sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha\end{cases}$$
to reduce A

$$A = m(\omega_0^2-\omega_n^2)\left[\cos\omega_n t\cos\delta_n - \sin\omega_n t\sin\delta_n\right] - \gamma\omega_n\left[\sin\omega_n t\cos\delta_n - \sin\delta_n\cos\omega_n t\right]$$

From notes,
$$\begin{cases}\cos\delta_n = \dfrac{m(\omega_0^2-\omega^2)}{D}\\ \sin\delta_n = \dfrac{\delta\omega_n}{D}\end{cases}$$
where (from before), $D = (m^2(\omega_0^2-\omega_n^2)^2 - \gamma^2\omega_n^2)^{1/2}$

So

$$\mathcal{L}x(t) = \sum_n \frac{a_n}{D}\Big[m(\omega_0^2-\omega_n^2)\Big[\cos\omega_n t\left(\frac{m(\omega_0^2-\omega_n^2)}{D}\right) - \sin\omega_n t\left(\frac{\delta\omega_n}{D}\right)\Big]$$
$$-\delta\omega_n\Big[\sin\omega_n t\left(\frac{m(\omega_0^2-\omega^2)}{D}\right) - \left(\frac{\delta\omega_n}{D}\right)\cos\omega_n t\Big]$$

[4] continued

Finally, we get: D^2 !!

$$\mathcal{L}\,x(t) = \sum_n \frac{a_n}{D^2}\left\{ \begin{array}{l} \left[m(\omega_0^2-\omega_n^2)+\delta^2\omega_n^2\right]\cos\omega_n t \\ + \left[\cancel{m(\omega_0^2-\omega_n^2)\,\delta\omega_n} - \cancel{m(\omega_0^2-\omega_n^2)\,\delta\omega_n}\right]\sin\omega_n t \end{array} \right\}$$

$$\mathcal{L}\,x(t) = \sum_n a_n \cos\omega_n t = F(t)$$

So $x(t)$ is a general solution to the differential equation.

[5] Let $\{\phi_n\}$, $\{E_n\}$ be eigenfunctions, values to TISE: $H\phi_n(x) = E_n\phi_n(x)$

Show that $\Psi(x,t) = \int dx'\, G(x,x',t)\,\Psi(x',0)$

where $G(x,x',t) = \sum_n \phi_n^*(x')\,\phi_n(x)\,e^{-iE_nt/\hbar}$

is a general solution for <u>any</u> initial $\Psi(x,0)$

We must show that $\Psi(x,t)$ satisfies the TDSE $\quad i\hbar\frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle$

First, we can expand $\Psi(x,0)$ in the $\{\phi_n\}$ eigenfunctions:

$$\Psi(x,0) = \sum_n a_n\,\phi_n(x)$$

Also, let $\quad \phi_n(x,t) = e^{-iE_nt/\hbar}\phi_n(x) \quad$ (time-evolved eigenstate)

$$\frac{\partial\Psi}{\partial t} = \frac{\partial}{\partial t}\int dx'\, G(x,x',t)\Psi(x',0) =$$

$$= \frac{\partial}{\partial t}\int dx' \underbrace{\left(\sum_n \phi_n^*(x')\phi_n(x)e^{-iE_nt/\hbar}\right)}_{G(x,x',t)}\underbrace{\left(\sum_m a_m\phi_m(x')\right)}_{\Psi(x',0)}$$

$$= \int dx' \sum_{n,m}\left[\left(\frac{-iE_n}{\hbar}\right)e^{-iE_nt/\hbar}\phi_n(x')\,\phi_n(x)\,a_m\phi_m(x')\right]$$

$$= \sum_{n,m}\left(\frac{-iE_n}{\hbar}\right)\phi_n(x)\,a_m\underbrace{\int dx'\,\phi_n^*(x')\,\phi_m(x')}_{=\delta_{nm} \text{ by orthonormality of states}}$$

$$= \sum_n a_n\left(-\frac{iE_n}{\hbar}\right)e^{-iE_nt/\hbar}\phi_n(x)$$

So, $$\boxed{i\hbar\frac{\partial\Psi}{\partial t} = \sum a_nE_n\,\phi_n(x,t)}$$

[5] continued

Now, $H\Psi(x,t)$

$$H\Psi(x,t) = H\int dx'\, G(x,x';t)\,\Psi(x';0)$$

↳ operates on $\underline{x}$ only!

$$= \int dx'\, H\Big(\sum_n \phi_n^*(x')\phi_n(x)\, e^{-iE_nt/\hbar}\Big)\Big(\sum_m a_m\phi_m(x')\Big)$$

$$= \int dx' \sum_{n,m} \phi_n^*(x')\cdot \underbrace{H\phi_n(x)}_{E_n\phi_n(x)}\, e^{-iE_nt/\hbar}\, a_m\phi_m(x')$$

$$= \sum_{n,m} E_n e^{-iE_nt/\hbar}\phi_n(x)\, a_m \underbrace{\int dx'\, \phi_n^*(x')\,\phi_m(x')}_{\delta_{nm}\ !!}$$

$$= \sum_n a_n E_n e^{-iE_nt/\hbar}\phi_n(x)$$

$$\boxed{H\Psi(x,t) = \sum_n a_n E_n \phi_n(x,t)}$$

So $\quad i\hbar\frac{\partial\Psi}{\partial t} = H\Psi \qquad$ and $|\Psi\rangle$ is

a general solution for $\underline{\text{any}}$ initial $\Psi(x,0)$.