P204A HW#4 SOLUTIONS

FALL, 2010

Physics 204A, Fall 2010, Problem Set 4

PTS

(20) [1.] In class we defined the "length" $|f|$ of $f(x)$ to be $|f|^2 = \int f(x)^2 dx$ (for real functions). Stone and Goldbart comment that, more generally, one can define the length $|f|$ by $|f|^p = \int f(x)^p dx$, and that the triangle inequality is satisfied for $1 \le p < \infty$. By considering $f(x) = 1$ and $g(x) = x$, show that the triangle inequality is indeed satisfied for $p = 2$ and $p = 3$, but is violated for $p = 1/2$. Use the interval $0 \le x \le 1$ for your domain for the functions.

(20) [2a.] Calculate the Fourier series for the sawtooth wave, $f(x) = x$, for x in $(-\pi, \pi)$ and $f(x + 2\pi) = f(x)$. (Note that in class we defined $f(x)$ on the interval $(0, 2\pi)$ but it is fine to use any interval.) Make an argument based on the appearance of the function for why some of the a_n and b_n vanish.

(20) [2b.] Calculate the sum of the Fourier series of [2a] using 4-, 6-, 8-, and 10- terms at $x/\pi = -1.00, -0.98, -0.96, \cdots, 0.00, 0.02, \ldots 1.00$. Plot your results.

(20) [3.] Calculate the Fourier series for $f(x) = x^2$, for x in $(-\pi, \pi)$ and $f(x + 2\pi) = f(x)$. By plugging in $x = 0$ and $x = \pi$ evaluate the sums $S_1 = \sum_{n=1}^{\infty} 1/n^2$ and $S_2 = \sum_{n=1}^{\infty} (-1)^{n+1}/n^2$. By using Parseval's theorem (see Stone and Goldbart), evaluate $S_3 = \sum_{n=1}^{\infty} 1/n^4$.

(20) [4.] In class we derived the Fourier series for the "square wave" function $f(x) = 1$ for $0 \le x < \pi$ and $f(x) = 0$ for $\pi \le x < 2\pi$. Write a program to evaluate the Fourier series for $x = \pi/8$, $\pi/4$, $3\pi/2$. How many terms do you need to include to get the correct $f(x)$ to an accuracy of 0.04?

[1] Let $\left\{ \begin{array}{l} f(x) = 1 \\ g(x) = x \end{array} \right\}$ $(f+g)(x) = 1 + x$ on $[0,1]$

Test "triangle inequality" for three different norms: $|f|^p = \int f(x)^p \, dx$

where $p = 2$, L^2 norm

$p = 3$, L^3 norm

$p = 1/2$, $L^{1/2}$ norm

For L^2 norm:

$|f|^2 = \int_0^1 1^2 \, dx = 1$ $|f| = 1$

$|g|^2 = \int_0^1 x^2 \, dx = 1/3$ $|g| = 1/\sqrt{3} = .577$

$|f+g|^2 = \int_0^1 (1+x)^2 = \int_0^1 (1 + 2x + x^2) = 1 + 1 + \frac{1}{3} = 7/3$

$|f+g| = \sqrt{\frac{7}{3}} = 1.527$

✓ $\boxed{|f| + |g| = 1.577 \geq 1.527 = |f+g|}$ OK

For L^3 norm:

$|f|^3 = 1$ $|g|^3 = 1/4$ $|g| = \sqrt[3]{1/4} = \boxed{.63}$

$|f+g|^3 = \int_0^1 (1+x)^3 \, dx = \int_0^1 (1 + 3x + 3x^2 + x^3) = 1 + \frac{3}{2} + 1 + \frac{1}{4}$

$= 3.75$

$|f+g| = \sqrt[3]{3.75} = 1.553$

✓ $\boxed{|f| + |g| = 1 + .63 \geq 1.553 = |f+g|}$ OK

[1] Continued

For $L^{1/2}$ norm:

$$|f|^{1/2} = \int_0^1 1^{1/2}\,dx = 1 \qquad |f| = 1$$

$$|g|^{1/2} = \int_0^1 x^{1/2}\,dx = \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{2}{3} \qquad |g| = \frac{4}{9} = .4444$$

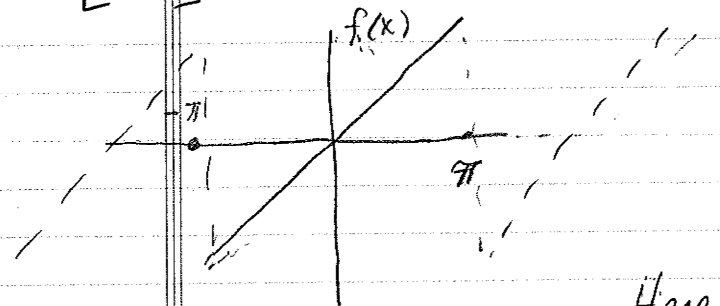
$$|f+g|^{1/2} = \int_0^1 (1+x)^{1/2}\,dx = \frac{2}{3}(1+x)^{3/2}\Big|_0^1 = \frac{2}{3}\left(2^{3/2}-1\right) \approx 1.218$$

$$|f+g| = (1.218)^2 = 1.4858$$

$$\boxed{|f|+|g| = 1.4444 \leq 1.4858 = |f+g|}$$

FAILS !!

[2a] Calculate Fourier series for $f(x) = x \quad -\pi < x < \pi$



$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=0}^{\infty} b_n \sin nx$$

Here, we expect $a_n = 0 \ \forall n$ since $f(x)$ is an odd function
(anti-symmetric)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx = \frac{1}{\pi}\left[-\frac{x}{n}\cos nx\Big|_{-\pi}^{\pi} + \cancelto{0}{\frac{1}{n}\int_{-\pi}^{\pi}\cos nx\,dx}\right]$$

$$b_n = \frac{-1}{\pi n}\left[\pi\cos n\pi + \pi\cos n\pi\right] = -\frac{2}{n}\cos n\pi$$

$$= \frac{2}{n}(-1)(-1)^n = \boxed{\frac{2}{n}(-1)^{n+1}}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x = 0$$

$$\Longrightarrow \boxed{f(x) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx}$$

HW#4 Problem 2b

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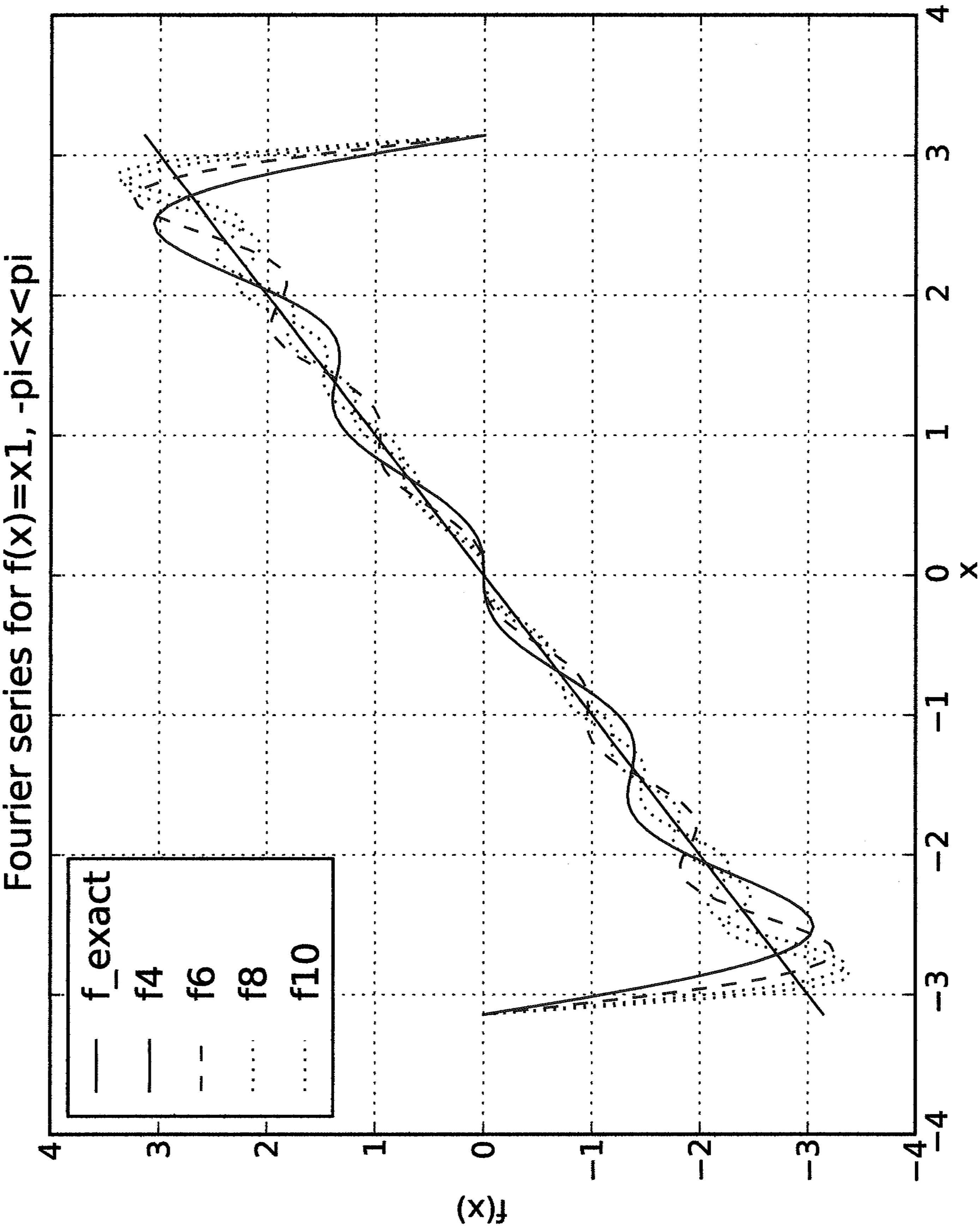


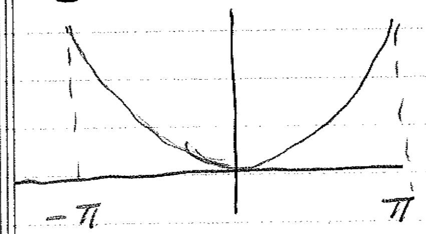
Hw #4 Problem 2b

5


```
#!/usr/bin/env python
#  Physics 204A HW#4 Problem 2b
#  For f(x) = 1 (0,pi), calculate FS for
#  x/pi = -1, -.98, -.96, ...,0, ..,1.0
#  Use 4,6,8,10 terms
import numpy as np
from math import pi, cos, sin, sqrt
from matplotlib.pylab import *
#
#   Build array of 'x' values to compute
#
x = linspace(-1,1,101) * pi
#
#   Function to compute fourier series for
#   variable number of terms (num_terms)
#      x is an array of input values
#
def FS_tot(num_terms, x):
    total = 0.0
    for n in range(num_terms+1):
        if n == 0:
            total += 0.0
        else:
            total += (2./n) * -cos(n*pi) * sin(n*x)

    return total
#
#  Allow FS_tot function to work on entire array of values
#
FS_tot_vect = vectorize(FS_tot)
#
#  Build FS arrays for 4, 6, 8 and 10 term series
#
f_exact = x
f_4 = FS_tot_vect(4,x)
f_6 = FS_tot_vect(6,x)
f_8 = FS_tot_vect(8,x)
f_10 = FS_tot_vect(10,x)
#
#   Plot FS on same diagram
#
plot(x,f_exact, 'b-', x, f_4, 'r-', x, f_6, 'g--', x, f_8,'r:', x, f_10,'m:')
title('Fourier series for f(x)=x1, -pi<x<pi')
l = legend(('f_exact','f4','f6','f8','f10'), loc ='upper left')
xlabel(' x ')
ylabel('f(x)')
grid()
savefig('FS_HW4_2b.pdf')
clf()
```

[3] Calculate FS for $f(x) = x^2 \quad -\pi < x < \pi$



$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

[Expect $b_n = 0$ $\forall n$ by symmetry of function]

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx = \frac{1}{\pi}\left[\cancelto{0}{\frac{x^2}{n}\sin nx\Big|_{-\pi}^{\pi}} - \int_{-\pi}^{\pi} \frac{2x}{n}\sin nx\,dx\right]$$

$$= -\frac{2}{\pi n}\left[-x\frac{\cos nx}{n}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos nx}{n}\,dx\right]$$

$$= \frac{2}{\pi n^2}\left[2\pi \cos n\pi\right] + \frac{1}{n^2}\cancelto{0}{\sin nx}\Big|_{-\pi}^{\pi}$$

$$a_n = \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 = \frac{1}{\pi}\frac{x^3}{3}\Big|_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{2\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$\Longrightarrow \boxed{f(x) = \underbrace{\frac{\pi^2}{3}}_{(a_0/2)} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx}$$

[3] continued

$$f(0) = \pi^2/3 + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(0) = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n = -\pi^2/3 \quad \text{(Now, multiply by } -1/4\text{)}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = \frac{\pi^2}{12}}$$

$$f(\pi) = \pi^2/3 + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \underbrace{\cos n\pi}_{(-1)^n} = \pi^2 \quad (x^2)$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^{2n} = 2/3\,\pi^2$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2}{12}\pi^2 = \frac{\pi^2}{6}}$$

<u>By Parseval's Thm</u>:

$$\|f\|^2 = \langle f|f\rangle = \sum_{n=1}^{\infty} |a_n|^2 \quad \text{(Stone \& Goldbart)}$$

where it was assumed that

$$f(x) = \sum a_n u_n(x) \quad \text{and} \quad \int u_m^*(x)\, u_n(x) = \delta_{mn}$$

$\{u_n(x)\}$ are an ortho<u>normal</u> basis

In our case, if we use $\{\cos nx, \sin nx\}$, basis is <u>not</u> normalized & we can't use "formula" directly

<u>Simplest solution is just to expand</u> $f(x)$ as follows

$$\int_{-\pi}^{\pi} [f(x)]^2 = \int_{-\pi}^{\pi} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx\right]^2$$

$$= \int_{-\pi}^{\pi} \left[\frac{a_0^2}{4} + \sum_{n=1}^{\infty} a_n^2 \cos^2 nx + \sum_{n=1}^{\infty} b_n^2 \sin^2 nx\right]$$

(all other terms will vanish due to orthogonality)

[3] continued

Integrating, we get:

$$\int_{-\pi}^{\pi} [f(x)]^2 = 2\pi \frac{a_0^2}{4} + \sum_{n=1}^{\infty} \pi a_n^2 + \sum_{n=1}^{\infty} \pi b_n^2$$

$$\left[\text{since } \int_{-\pi}^{\pi} \cos^2 nx = \int_{-\pi}^{\pi} \sin^2 nx = \pi\right]$$

$$\int_{-\pi}^{\pi} x^4 = \pi \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + \cancel{b_n^2}^{\,0})\right]$$

OR

$$\left.\frac{x^5}{5}\right|_{-\pi}^{\pi} = \pi \left[\frac{1}{2}\left(\frac{4\pi^4}{9}\right) + \sum_{n=1}^{\infty} \frac{16}{n^4}(-1)^{2n}\right]$$

$$\frac{2\pi^5}{5} = \frac{2\pi^5}{9} + \pi \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{2\pi^5}{16\pi}\left[\frac{1}{5} - \frac{1}{9}\right] = \frac{\pi^4}{8}\left[\frac{9-5}{45}\right]$$

$$\Longrightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}$$

Here, we applied same logic that went into proof of Parseval's theorem

[4] From class notes, we derived

Fourier series for $\begin{cases} f(x) = 1 & 0 \le x < \pi \\ f(x) = 0 & \pi \le x < 2\pi \end{cases}$

as $$\boxed{f(x) = \frac{1}{2} + \sum_{n \text{ odd}} \frac{2}{n\pi} \sin nx}$$

Program to evaluate number of terms needed for accuracy to <u>.04</u> is attached

<u>Note</u>: I counted <u>a_0</u> as a term and then all contributing (non-zero) terms thereafter in the <u>count</u>.

HW # 4   Problem # 4   10

IDLE_tmp_dcsdjp

```
Python 2.6.5 |EPD 6.2-2 (32-bit)| (r265:79063, May  7 2010, 13:28:19) [MSC v.1500 32
bit (Intel)] on win32
Type "copyright", "credits" or "license()" for more information.

    ****************************************************************
    Personal firewall software may warn about the connection IDLE
    makes to its subprocess using this computer's internal loopback
    interface.  This connection is not visible on any external
    interface and no data is sent to or received from the Internet.
    ****************************************************************

IDLE 2.6.5
>>> ================================ RESTART ================================
>>>
 P204A HW4 Problem 4:
     For f(x) = 1 (0,pi), calculate FS for
     x  = pi/8, pi/4, 3pi/2. Calculate number
     of terms needed for accuracy of 0.04
  Note: a(0) fourier term included in count !!
For x = 0.392699081699, f(x) = 1 :
   Fourier series = 0.743623839601, number of terms = 2
   Fourier series = 0.939677165495, number of terms = 3
   Fourier series = 1.05730916103, number of terms = 4
   Fourier series = 1.09211256669, number of terms = 5
   Fourier series = 1.06504325118, number of terms = 6
   Fourier series = 1.0115741623, number of terms = 7
       7 fourier terms are required for .04 accuracy

For x = 0.785398163397, f(x) = 1 :
   Fourier series = 0.950158158079, number of terms = 2
   Fourier series = 1.10021087744, number of terms = 3
   Fourier series = 1.01017924582, number of terms = 4
       4 fourier terms are required for .04 accuracy

For x = 4.71238898038, f(x) = 0 :
   Fourier series = -0.136619772368, number of terms = 2
   Fourier series = 0.0755868184216, number of terms = 3
   Fourier series = -0.0517371360519, number of terms = 4
   Fourier series = 0.0392085457149, number of terms = 5
       5 fourier terms are required for .04 accuracy

>>>
```

```
#!/usr/bin/env python
#  Physics 204A HW#4 Problem 4
#  For f(x) = 1 (0,pi), calculate FS for
#  x  = pi/8, pi/4, 3pi/2
#    Calculate number of terms needed to
#    accuracy of 0.04
import numpy as np
from math import pi, cos, sin, sqrt
from matplotlib.pylab import *
print ' P204A HW4 Problem 4:'
print '      For f(x) = 1 (0,pi), calculate FS for'
print '      x  = pi/8, pi/4, 3pi/2. Calculate number'
print '      of terms needed for accuracy of 0.04'
print '  Note: a(0) fourier term included in count !!'
#
#   Build list of 'x' and 'f(x)' values a to test
#
x_values = [pi/8, pi/4, 3*pi/2]
f_values = [ 1 , 1, 0]
min_acc  = .04

for i in range(len(x_values)):
    x = x_values[i]
    f_actual = f_values[i]
    print 'For x = %s, f(x) = % s : ' % (x,f_actual)
    f_FS = .5                    #  Value of a(0)
    n = 1
    num_terms = 1                # Include a(0) term in count
#  Loop by adding fourier terms until desired
#    accuracy is reached
    while abs(f_FS - f_actual)> min_acc :
        num_terms += 1
        f_FS += (2./(n*pi)) * sin(n*x)
        print '    Fourier series = %s, number of terms = %d' % (f_FS,num_terms)
        n = n + 2                #  Odd values only
#  Print out all results
#
    print '         %d fourier terms are required for .04 accuracy' % (num_terms)
    print'\n'
```