P 204A Fall, 2010 (1)

SOLUTIONS HW # 3

Physics 204A, Fall 2010, Problem Set 3

10 pts

[1a.] An operator $\mathcal{P}$ is said to be a "projection" if $\mathcal{P} = \mathcal{P}^2$. Prove the eigenvalues of a projection operator must be $\lambda = 0, 1$.

[1b.] A unit vector $|w\rangle$ has components w_i in a given orthonormal basis in a three dimensional space. That is, $|w\rangle = w_1|e_1\rangle + w_2|e_2\rangle + w_3|e_3\rangle$ with $\sum w_i^2 = 1$. Write the matrix (in the basis $|e_i\rangle$) representing the operator which projects any other vector $|v\rangle$ onto the plane perpendicular to $|w\rangle$. Show it obeys the result of [1a].

20 pts

[2.] Redo the coupled mass-spring problem in class, but chose the masses to alternate: $M_l = M_A$ for l even and $M_l = M_B$ for l odd. *Hint:* It is fine to assume the same time dependence for $x_l(t) = v_l\, e^{i\omega t}$ as in class. However, the spatial dependence $v_l = v_0\, e^{iql}$ is no longer quite right. Can you think of a relatively small variation of this *ansatz* which recognizes that $M_A \neq M_B$, but still takes advantage of the fact that $M_1 = M_3 = M_5 = \cdots = M_B$ and $M_2 = M_4 = M_6 = \cdots = M_A$? Put another way, the system still has translation invariance, but you have to consider a "unit" consisting of a pair of masses M_A, M_B.

20 pts

[3.] Show that

$$x_l(t) = v_0(-1)^l\, \frac{1-\epsilon}{1+\epsilon}\, e^{i\omega t}$$
$$\omega^2 = \frac{4K}{M} \frac{1}{1-\epsilon^2}$$

is a solution of the problem of an infinite set of vibrating masses M connected by springs K and with a light defect $M' = M(1-\epsilon)$ at $l = 0$. Does this functional form make sense as $\epsilon \to 0$ and as $\epsilon \to 1$?

Comment: To do problems [4a], and [4b] below, you may want to keep M as part of the matrix containing the spring constant K, since it is no longer constant. Is your matrix symmetric? (If not, be careful you do not use a numerical routine for diagonalization that assumes symmetric.) You can avoid these issues by doing an alternate problem with a defect spring instead of a defect mass if you prefer.

30 pts

[4a.] Solve for all the normal modes of a collection of $N = 128$ masses M connected by springs K. Show that you agree with the solution in class, *e.g.* verify 4-5 of your list of eigenvalues are correct. Also show all the participation ratios are "large", *i.e.* within a factor of 2 or so of N. *Note:* There is a subtlety here because all but two of the eigenvalues are doubly degenerate: q and $2\pi - q$ have the same ω. In such a case the eigenvectors can be arbitrary linear combinations.

[4b.] Redo [4a] with a single defect mass $M(1-\epsilon)$. Verify your result agrees with [3]. Plot one of the delocalized eigenvectors and also the localized eigenvector for $\epsilon = 0.1$. Show that one of your participation ratios is "small",

(2)

20 pts

[5.] In class we showed in general that stochastic matrices have eigenvalues which obey $|\lambda| \leq 1$, and that $\lambda = 1$ is one of the eigenvalues. Show this is the case for the specific matrix

$$\begin{matrix} \frac{1}{3} & 0 & \frac{1}{2} \\ \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \end{matrix}$$

Determine the left and right eigenvectors with eigenvalue $\lambda = 1$. Are they the same?

P204A F2010 HW #3 SOLUTIONS

1a If $|v\rangle$ is an eigenvector of P then

$$P|v\rangle = \lambda|v\rangle$$

And hence $P^2|v\rangle = \lambda P|v\rangle = \lambda^2|v\rangle$

But we also know $P^2 = P$ so $P^2|v\rangle = \lambda|v\rangle$

So $\lambda|v\rangle = \lambda^2|v\rangle$

which can only hold if $\lambda = 0, 1$.

1b Pretty clearly we want to define

$$P\vec{v} = \vec{v} - (\vec{v}\cdot\vec{w})\vec{w}$$

since this subtracts of part of $\vec{v}$ which is parallel to $\vec{w}$. Written out in components

$$P\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} - \begin{pmatrix} (v_1w_1 + v_2w_2 + v_3w_3)\, w_1 \\ \text{''} \quad w_2 \\ \text{''} \quad w_3 \end{pmatrix}$$

$$= \begin{bmatrix} 1-w_1^2 & -w_1w_2 & -w_1w_3 \\ -w_1w_2 & 1-w_2^2 & -w_2w_3 \\ -w_1w_3 & -w_2w_3 & 1-w_3^2 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

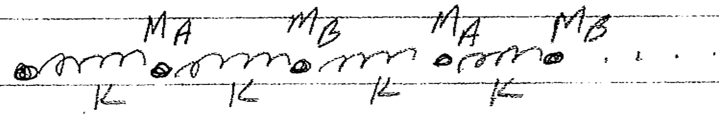
At this point we could form $|P - \lambda I| = 0$ and get λ. That might be messy. Instead "guess" eigenvectors eg $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ clearly obeys $P\vec{w} = \vec{0}$ assuming $|\vec{w}| = 1$

Also $\begin{pmatrix} w_2 \\ -w_1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} w_3 \\ 0 \\ -w_1 \end{pmatrix}$ are eigenvectors of eigenvalue 1. We guessed these because they are $\perp$ to $\vec{w}$.

2 Mass-spring problem with alternating masses:

$M_\ell = M_A$ for ℓ odd

$\quad = M_B$ for ℓ odd



Equations of motion are similar

① $\quad M_A \ddot{x}_\ell = -K(x_\ell - x_{\ell+1}) - K(x_\ell - x_{\ell-1}) \quad \ell$ odd

② $\quad M_B \ddot{x}_\ell = -K(x_\ell - x_{\ell+1}) - K(x_\ell - x_{\ell-1}) \quad \ell$ even

However, this time we vary the amplitude of the motion for the two masses in the ansatz:

$$x_\ell(t) = A_0\, e^{iq\ell} e^{i\omega t} \qquad \ell \text{ odd}$$

$$x_\ell(t) = B_0\, e^{iq\ell} e^{i\omega t} \qquad \ell \text{ even}$$

Substituting these into ① and ② and eliminating common factors ($e^{iq\ell}e^{i\omega t}$), we obtain:

ℓ odd: $\quad -\omega^2 M_A A_0 = -2KA_0 + KB_0(e^{iq} + e^{-iq})$

ℓ even: $\quad -\omega^2 M_B B_0 = -2KB_0 + KA_0(e^{iq} + e^{-iq})$

Let $e^{iq} + e^{iq} = 2\cos q$ and put into matrix format:

$$\begin{pmatrix} -\omega^2 M_A + 2K & -2K\cos q \\ -2K\cos q & -\omega^2 M_B + 2K \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = 0$$

2 cont'd

Only non-trivial solution occurs when determinant of matrix vanishes or:

$$(-\omega^2 M_A + 2K)(-\omega^2 M_B + 2K) - 4K^2 \cos^2 q = 0$$

or $$\boxed{M_A M_B \omega^4 - 2K(M_A + M_B)\omega^2 + 4K^2(1-\cos^2 q) = 0}$$

Solution are roots:

$$\omega^2 = \frac{2K(M_A+M_B) \pm \sqrt{4K^2(M_A+M_B)^2 - 16 M_A M_B K^2 \sin^2 q}}{2 M_A M_B}$$

$$\boxed{(A)\quad \omega^2 = \frac{K(M_A+M_B) \pm K\sqrt{(M_A+M_B)^2 - 4M_A M_B \sin^2 q}}{M_A M_B}}$$

Note that for periodic boundary conditions (and for translational invariance), both odd & even masses have 2X periodicity so $q \approx \frac{2\pi}{N} n$

Equation (A) can describe the frequency behavior of "phonons" or lattice vibrations in a diatomic crystal (two different atoms)

(1) Minus sign branch in (A) above begins at the point $\omega = 0$ $q = 0$. As q increases, ω increases (for both negative & positive q) until $q = \pi/2$ (Acoustic branch)

(2) Plus sign branch in (1) stay at high frequency (ω) for all values of q (Optical branch)

3. We have a special eqn for $l=0$ since $M' \neq M$

$l=0$ $\quad M'\ddot{x}_0 = -k(x_0 - x_1) - k(x_0 - x_{-1})$

and the same eqn for all $l \neq 0$

$$M\ddot{x}_l = -k(x_l - x_{l+1}) - k(x_l - x_{l-1})$$

Plugging in our brilliant guess to the top eqn and cancelling $V_0 e^{+i\omega t}$ yields

$$-M'\omega^2 \stackrel{?}{=} -k\left[1-(-1)\frac{1-\epsilon}{1+\epsilon}\right] - k\left[1-(-1)\frac{1-\epsilon}{1+\epsilon}\right]$$

$$= -2k\left[\frac{1+\epsilon+1-\epsilon}{1+\epsilon}\right] = -2k\left[\frac{2}{1+\epsilon}\right]$$

Thus we need $\quad \omega^2 = \dfrac{4k}{(1+\epsilon)M'} = \dfrac{4k}{(1-\epsilon^2)M}$

which is indeed the ω claimed.

Trying out the bottom eqn

$$-M\omega^2 \stackrel{?}{=} -k\left[1-\frac{1-\epsilon}{1+\epsilon}(-1)\right] - k\left[1-\left(\frac{1+\epsilon}{1-\epsilon}\right)(-1)\right]$$

$$= -k\left[1+\frac{1-\epsilon}{1+\epsilon}+1+\frac{1+\epsilon}{1-\epsilon}\right]$$

$$= -k\,\Big[\underbrace{1-\epsilon^2+(1-\epsilon)^2+1-\epsilon^2+(1+\epsilon)^2}_{2-2\epsilon^2+1-2\epsilon+\epsilon^2+1+\cancel{2\epsilon}+\epsilon^2}\Big]\Big/(1+\epsilon)(1-\epsilon)$$

So we are again okay.

$$= -4k/(1-\epsilon^2)$$

3 cont'd

$$N' = N(1-\epsilon)$$

$$v_\ell = v_0(-1)^\ell \left(\frac{1-\epsilon}{1+\epsilon}\right)^\ell$$

To show that form makes sense as $\epsilon \to 0$ or $\epsilon \to 1$

Normalizing: (infinite chain)

$$\sum_{\ell=-\infty}^{\infty} v_0^2 \left(\frac{1-\epsilon}{1+\epsilon}\right)^{2\ell} = 1$$

$$\sum_{-\infty}^{\infty} r^\ell = 1 + 2\sum_1^\infty r^\ell = 1 + 2r\sum_0^\infty r^\ell = 1 + 2r/1-r = \frac{1+r}{1-r}$$

Here $1+r = 1 + \left(\frac{1-\epsilon}{1+\epsilon}\right)^2 = \frac{1+2\epsilon+\epsilon^2+1-2\epsilon+\epsilon^2}{(1+\epsilon)^2} = 2\frac{1+\epsilon^2}{(1+\epsilon)^2}$

$$1-r = 1 - \left(\frac{1-\epsilon}{1+\epsilon}\right)^2 = \frac{1+2\epsilon+\epsilon^2-1+2\epsilon-\epsilon^2}{(1+\epsilon)^2} = \frac{4\epsilon}{(1+\epsilon)^2}$$

$$1+r/1-r = 1+\epsilon^2/2\epsilon$$

and so $v_0^2 = 2\epsilon/1+\epsilon^2$ to normalize

$$P^{-1} = \sum v_\ell^4 = \sum_\ell v_0^4 \left(\frac{1-\epsilon}{1+\epsilon}\right)^{4\ell} = v_0^4 \frac{1+r^2}{1-r^2}$$

$$= \left(\frac{1-r}{1+r}\right)^2 \frac{1+r^2}{1-r^2} = \frac{(1-r)(1+r^2)}{(1+r)^3} \qquad r = \left(\frac{1-\epsilon}{1+\epsilon}\right)^2$$

$r \to 0$ $\quad P^{-1} \Rightarrow 1 \quad P \Rightarrow 1 \quad W$ superlocalized

$r \to 1$ $\quad P^{-1} \Rightarrow 0 \quad P \Rightarrow \infty \quad W$ extended

5. Compute eigenvalues $\begin{vmatrix} 1/3-\lambda & 0 & 1/2 \\ 1/3 & 1/2-\lambda & 1/6 \\ 1/3 & 1/2 & 1/3-\lambda \end{vmatrix} = 0$

$$(1/3-\lambda)\left[(1/2-\lambda)(1/3-\lambda) - 1/12\right] + 1/2\left[1/6 - 1/3(1/2-\lambda)\right] = 0$$

$$(1/3-\lambda)\left[\lambda^2 - 5/6\,\lambda + 1/12\right] + 1/2\left[+\frac{\lambda}{3}\right] = 0$$

$$\begin{aligned} &-\lambda^3 + 5/6\,\lambda^2 - 1/12\,\lambda \\ &\qquad + 1/3\,\lambda^2 - 5/18\,\lambda + 1/36 \\ &\qquad\qquad + 1/6\,\lambda \\ &-\lambda^3 + 7/6\,\lambda^2 - 7/36\,\lambda + 1/36 = 0 \end{aligned}$$

$\boxed{\text{Clearly } \lambda = 1 \text{ is a soln!}}$ $\quad -1 + 7/6 - 7/36 + 1/36 = 0$

To find others, do "synthetic division" (I did not ask you to do this...)

$$\begin{array}{r} -\lambda^2 + 1/6\,\lambda - 1/36 \\ (\lambda-1)\ \overline{\big)\ -\lambda^3 + 7/6\,\lambda^2 - 7/36\,\lambda + 1/36} \\ \underline{-\lambda^3 + \lambda^2} \qquad\qquad\qquad \\ 1/6\,\lambda^2 - 7/36\,\lambda \qquad \\ \underline{1/6\,\lambda^2 - 1/6\,\lambda} \qquad \\ -1/36\,\lambda + 1/36 \end{array}$$

$$-(\lambda-1)(\lambda^2 - 1/6\,\lambda + 1/36) = 0$$

$$\lambda = \frac{1/6 \pm \sqrt{1/36 - 4/36}}{2}$$ $\Leftarrow$ Complex! But that's okay because our matrix was not symmetric. Nothing guaranteed

$$\Rightarrow \boxed{\lambda = \frac{1}{12}(1 \pm i\sqrt{3})}$$

5 cont'd   As discussed in class, the left eigenvector is:

$$\overline{V}_L = (1\ 1\ 1)\begin{pmatrix} 1/3 & 0 & 1/2 \\ 1/3 & 1/2 & 1/6 \\ 1/3 & 1/2 & 1/3 \end{pmatrix} = (1\ 1\ 1)$$

The right one.

$$\overline{V}_R: \quad \begin{pmatrix} -2/3 & 0 & 1/2 \\ 1/3 & -1/2 & 1/6 \\ 1/3 & 1/2 & -2/3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Top Eqn $-4v_1 + 3v_3 = 0$ $v_1 = \frac{3}{4}v_3$

Combining 2nd / 3rd Eqns $-v_2 + 5/6\, v_3 = 0$ $v_2 = 5/6\, v_3$

$$\boxed{\overline{V}_R = \begin{pmatrix} 3/4 \\ 5/6 \\ 1 \end{pmatrix} \longrightarrow \frac{1}{\sqrt{325}}\begin{pmatrix} 9 \\ 10 \\ 12 \end{pmatrix} = \frac{1}{5\sqrt{13}}\begin{pmatrix} 9 \\ 10 \\ 12 \end{pmatrix}}$$

obviously not the same as (1 1 1)!

4 a b Program Uniform Mass

11

```
IDLE_tmp_rdoqfs
Python 2.6.5 |EPD 6.2-2 (32-bit)| (r265:79063, May  7 2010, 13:28:19) [MSC v.1500 32
bit (Intel)] on win32
Type "copyright", "credits" or "license()" for more information.

    ****************************************************************
    Personal firewall software may warn about the connection IDLE
    makes to its subprocess using this computer's internal loopback
    interface.  This connection is not visible on any external
    interface and no data is sent to or received from the Internet.
    ****************************************************************

IDLE 2.6.5      ==== No Subprocess ====
>>>
Calculated Eigenvalues from matrix
[  2.83043535e-16   2.40908759e-03   2.40908759e-03   9.63054666e-03
   9.63054666e-03   2.16469801e-02   2.16469801e-02   3.84294392e-02
   3.84294392e-02   5.99374936e-02   5.99374936e-02   8.61193285e-02
   8.61193285e-02   1.16911870e-01   1.16911870e-01   1.52240935e-01
   1.52240935e-01   1.92021414e-01   1.92021414e-01   2.36157471e-01
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   7.31213432e-01   8.08601391e-01   8.08601391e-01   8.88859534e-01
   8.88859534e-01   9.71794512e-01   9.71794512e-01   1.05720653e+00
   1.05720653e+00   1.14488981e+00   1.14488981e+00   1.23463314e+00
   1.23463314e+00   1.32622029e+00   1.32622029e+00   1.41943065e+00
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   1.60981936e+00   1.70653905e+00   1.70653905e+00   1.80396572e+00
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   2.00000000e+00   2.09813535e+00   2.09813535e+00   2.19603428e+00
   2.19603428e+00   2.29346095e+00   2.29346095e+00   2.39018064e+00
   2.39018064e+00   2.48596036e+00   2.48596036e+00   2.58056935e+00
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   2.76536686e+00   2.85511019e+00   2.85511019e+00   2.94279347e+00
   2.94279347e+00   3.02820549e+00   3.02820549e+00   3.11114047e+00
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   3.66293922e+00   3.71545722e+00   3.71545722e+00   3.76384253e+00
   3.76384253e+00   3.80797859e+00   3.80797859e+00   3.84775907e+00
   3.84775907e+00   3.88308813e+00   3.88308813e+00   3.91388067e+00
   3.91388067e+00   3.94006251e+00   3.94006251e+00   3.96157056e+00
   3.96157056e+00   3.97835302e+00   3.97835302e+00   3.99036945e+00
   3.99036945e+00   3.99759091e+00   3.99759091e+00   4.00000000e+00]
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   3.84294392e-02   5.99374936e-02   5.99374936e-02   8.61193285e-02
   8.61193285e-02   1.16911870e-01   1.16911870e-01   1.52240935e-01
   1.52240935e-01   1.92021414e-01   1.92021414e-01   2.36157471e-01
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   3.37060775e-01   3.93584937e-01   3.93584937e-01   4.53979093e-01
   4.53979093e-01   5.18097749e-01   5.18097749e-01   5.85786438e-01
   5.85786438e-01   6.56882090e-01   6.56882090e-01   7.31213432e-01
   7.31213432e-01   8.08601391e-01   8.08601391e-01   8.88859534e-01
   8.88859534e-01   9.71794512e-01   9.71794512e-01   1.05720653e+00
   1.05720653e+00   1.14488981e+00   1.14488981e+00   1.23463314e+00
   1.23463314e+00   1.32622029e+00   1.32622029e+00   1.41943065e+00
   1.41943065e+00   1.51403964e+00   1.51403964e+00   1.60981936e+00
```

Eigenvalues

match well

$$\omega^2 = \frac{2K}{M}(1-\cos q)$$

$$q = \frac{2\pi n}{N}$$

$N = 128$

$n = 1, 2, \ldots, 128$

IDLE_tmp_rdoqfs

```
   1.60981936e+00   1.70653905e+00   1.70653905e+00   1.80396572e+00
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   2.00000000e+00   2.09813535e+00   2.09813535e+00   2.19603428e+00
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   2.76536686e+00   2.85511019e+00   2.85511019e+00   2.94279347e+00
   2.94279347e+00   3.02820549e+00   3.02820549e+00   3.11114047e+00
   3.11114047e+00   3.19139861e+00   3.19139861e+00   3.26878657e+00
   3.26878657e+00   3.34311791e+00   3.34311791e+00   3.41421356e+00
   3.41421356e+00   3.48190225e+00   3.48190225e+00   3.54602091e+00
   3.54602091e+00   3.60641506e+00   3.60641506e+00   3.66293922e+00
   3.66293922e+00   3.71545722e+00   3.71545722e+00   3.76384253e+00
   3.76384253e+00   3.80797859e+00   3.80797859e+00   3.84775907e+00
   3.84775907e+00   3.88308813e+00   3.88308813e+00   3.91388067e+00
   3.91388067e+00   3.94006251e+00   3.94006251e+00   3.96157056e+00
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   3.99036945e+00   3.99759091e+00   3.99759091e+00   4.00000000e+00]
>>>
```
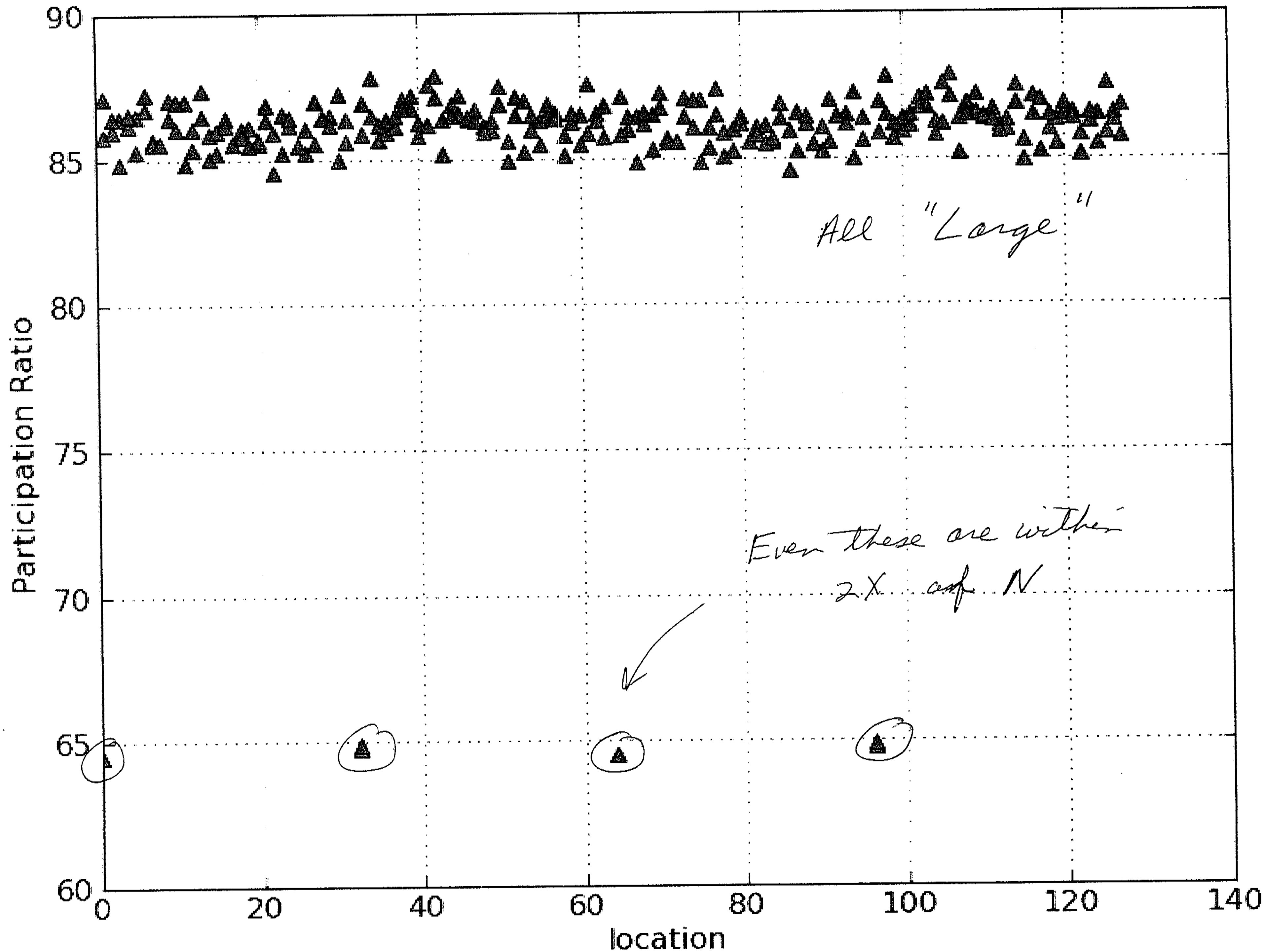
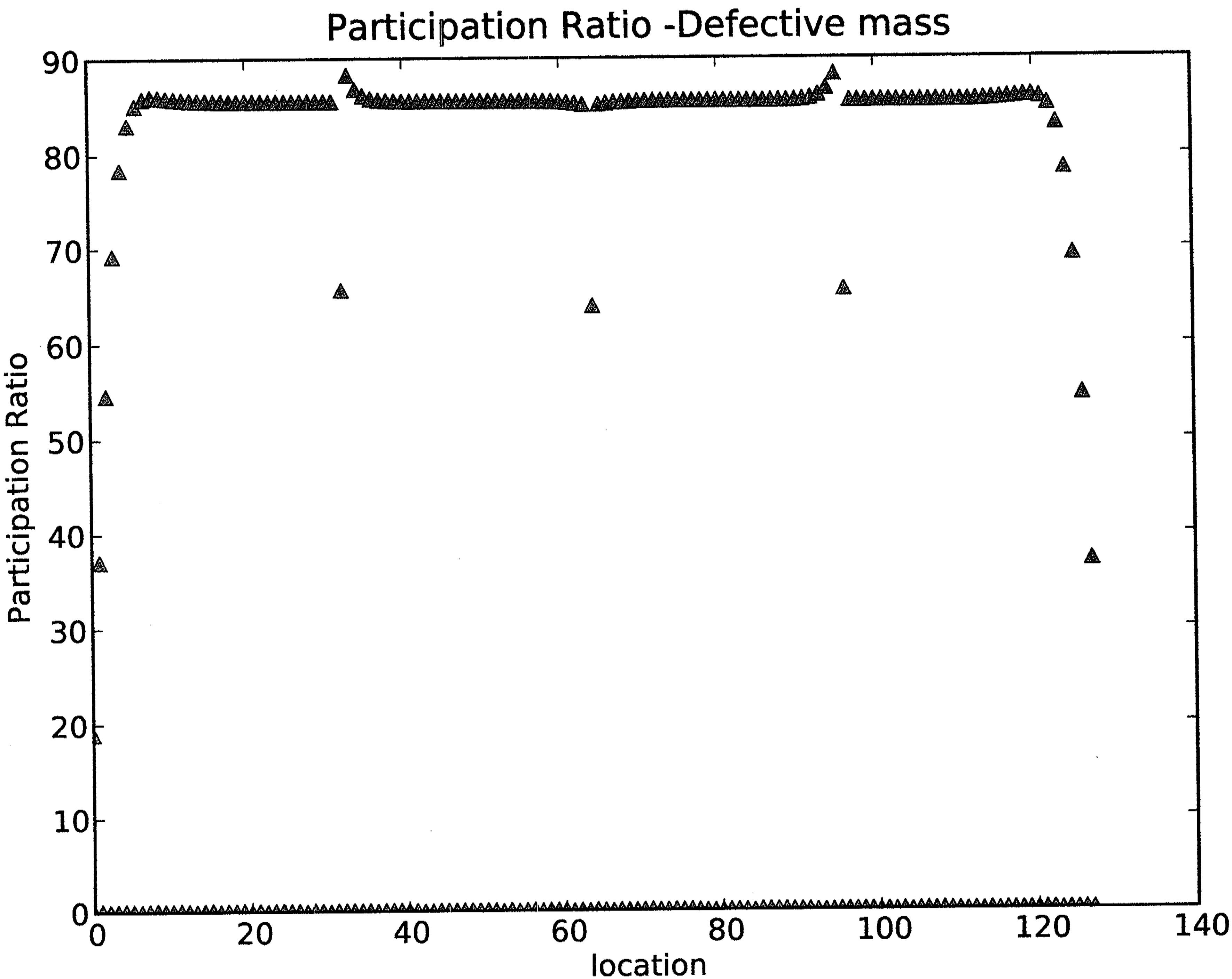
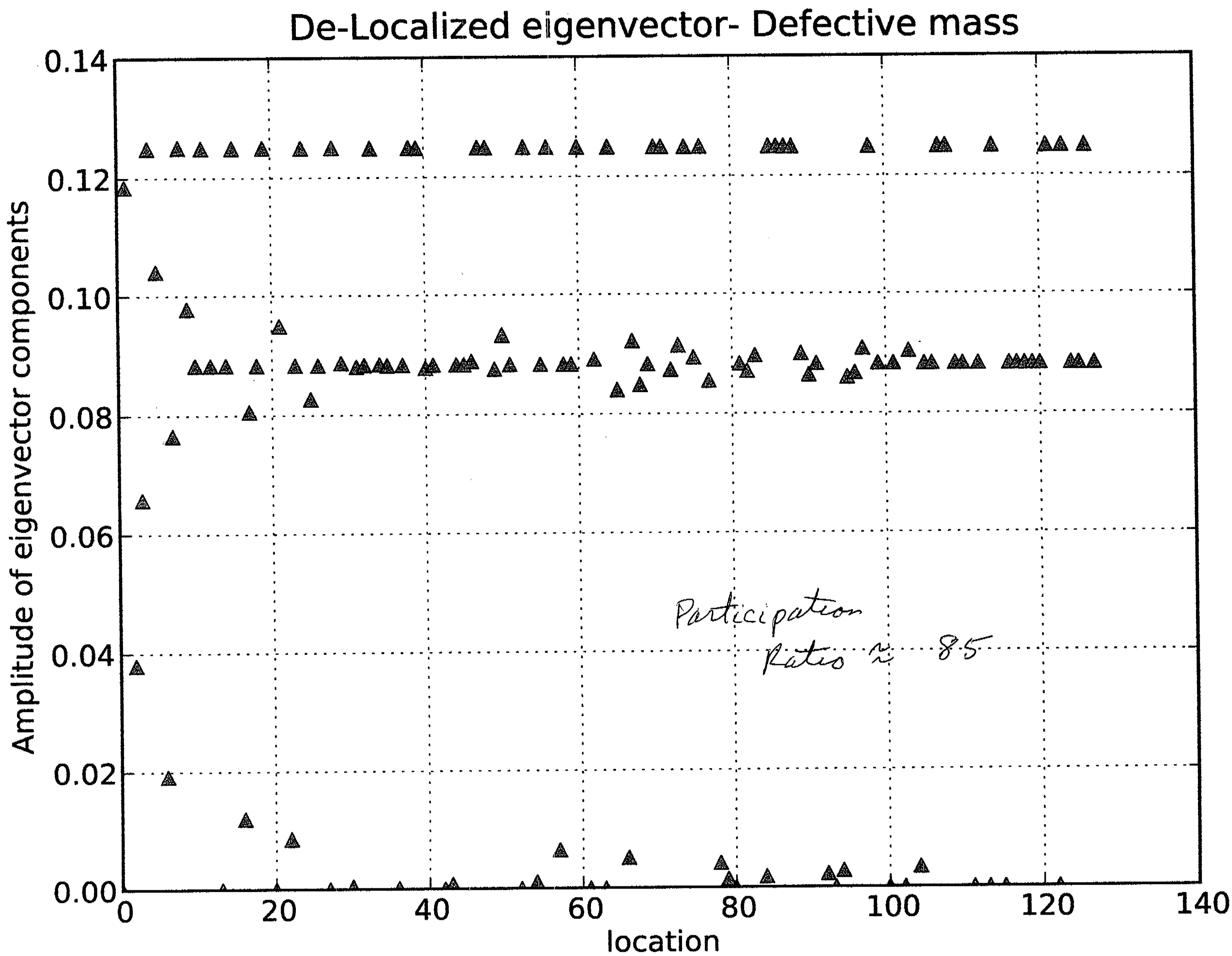

Uniform Mass

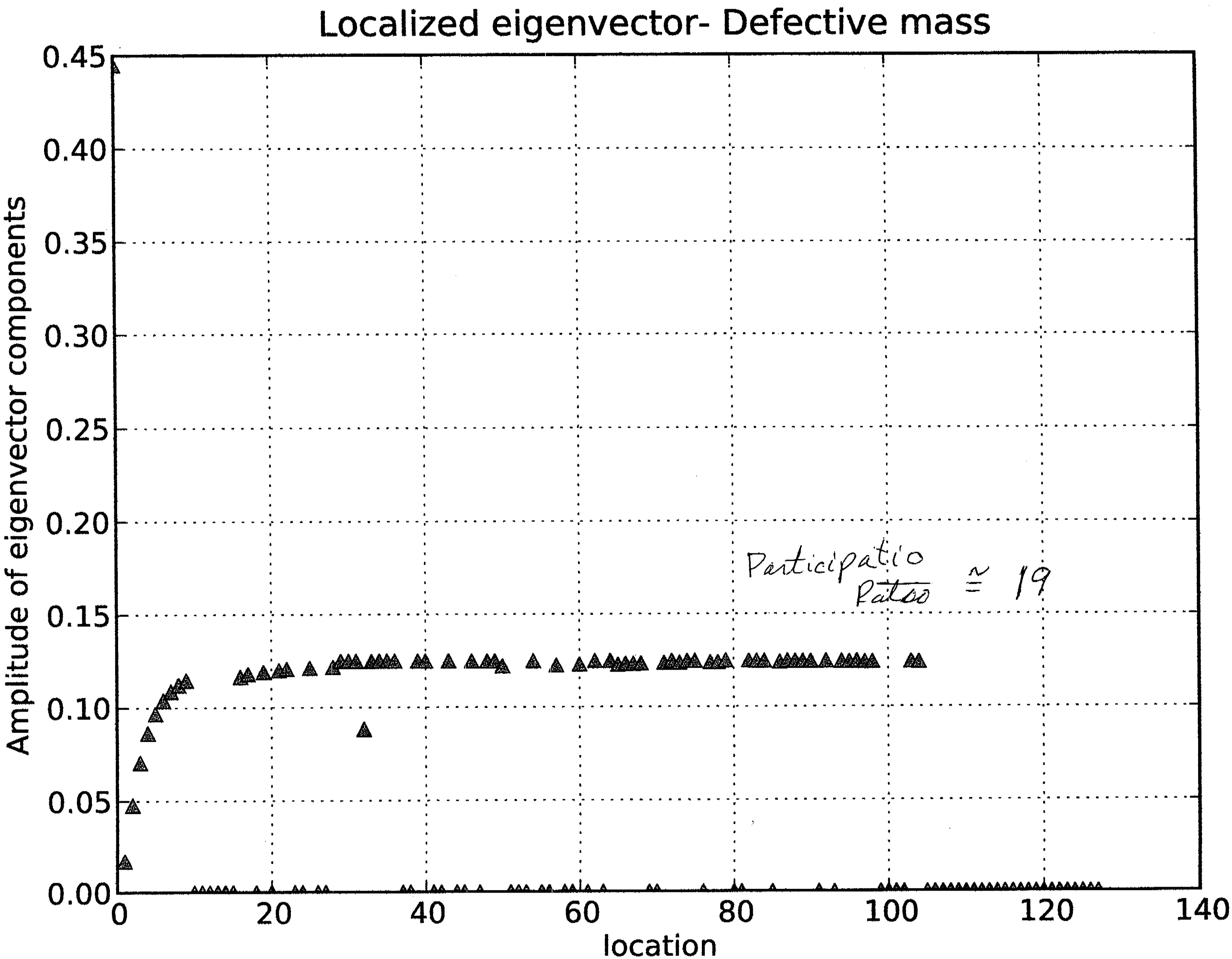


Defective Mass


```
IDLE_tmp_payhdi
Python 2.6.5 |EPD 6.2-2 (32-bit)| (r265:79063, May  7 2010, 13:28:19) [MSC v.1500 32
bit (Intel)] on win32
Type "copyright", "credits" or "license()" for more information.

    ****************************************************************
    Personal firewall software may warn about the connection IDLE
    makes to its subprocess using this computer's internal loopback
    interface.  This connection is not visible on any external
    interface and no data is sent to or received from the Internet.
    ****************************************************************

IDLE 2.6.5      ==== No Subprocess ====
>>>
 Defective mass at l =0 location case

 Calculated Eigenvalues from matrix
[  1.20566259e-16   2.40908759e-03   2.41285620e-03   9.63054666e-03
   9.64561193e-03   2.16469801e-02   2.16808426e-02   3.84294392e-02
   3.84895539e-02   5.99374936e-02   6.00312516e-02   8.61193285e-02
   8.62540391e-02   1.16911870e-01   1.17094742e-01   1.52240935e-01
   1.52479063e-01   1.92021414e-01   1.92321754e-01   2.36157471e-01
   2.36526832e-01   2.84542780e-01   2.84987799e-01   3.37060775e-01
   3.37587908e-01   3.93584937e-01   3.94200437e-01   4.53979093e-01
   4.54689000e-01   5.18097749e-01   5.18907871e-01   5.85786438e-01
   5.86702339e-01   6.56882090e-01   6.57909077e-01   7.31213432e-01
   7.32356539e-01   8.08601391e-01   8.09865368e-01   8.88859534e-01
   8.90248835e-01   9.71794512e-01   9.73313284e-01   1.05720653e+00
   1.05885860e+00   1.14488981e+00   1.14667869e+00   1.23463314e+00
   1.23656197e+00   1.32622029e+00   1.32829191e+00   1.41943065e+00
   1.42164751e+00   1.51403964e+00   1.51640386e+00   1.60981936e+00
   1.61233266e+00   1.70653905e+00   1.70920280e+00   1.80396572e+00
   1.80678091e+00   1.90186465e+00   1.90483187e+00   2.00000000e+00
   2.00311947e+00   2.09813535e+00   2.10140690e+00   2.19603428e+00
   2.19945733e+00   2.29346095e+00   2.29703454e+00   2.39018064e+00
   2.39390341e+00   2.48596036e+00   2.48983052e+00   2.58056935e+00
   2.58458474e+00   2.67377971e+00   2.67793772e+00   2.76536686e+00
   2.76966450e+00   2.85511019e+00   2.85954402e+00   2.94279347e+00
   2.94735962e+00   3.02820549e+00   3.03289963e+00   3.11114047e+00
   3.11595781e+00   3.19139861e+00   3.19633386e+00   3.26878657e+00
   3.27383387e+00   3.34311791e+00   3.34827082e+00   3.41421356e+00
   3.41946494e+00   3.48190225e+00   3.48724415e+00   3.54602091e+00
   3.55144441e+00   3.60641506e+00   3.61190999e+00   3.66293922e+00
   3.66849381e+00   3.71545722e+00   3.72105748e+00   3.76384253e+00
   3.76947140e+00   3.80797859e+00   3.81361446e+00   3.84775907e+00
   3.85337343e+00   3.88308813e+00   3.88864154e+00   3.91388067e+00
   3.91931566e+00   3.94006251e+00   3.94529080e+00   3.96157056e+00
   3.96644967e+00   3.97835302e+00   3.98264466e+00   3.99036945e+00
   3.99367531e+00   3.99759091e+00   3.99929192e+00   4.04040404e+00]
                                        match
 Theoretical frq**2
4.0404040404

 Localized eigenvector & participation ratio   <==  Localized
0 18.9698591493

 De-Localized eigenvector & participation ratio   <==  De-localized
80 85.5506591682
>>>
```


Participation Ratio -Defective mass
Participation Ratio
location
0
10
20
30
40
50
60
70
80
90
0
20
40
60
80
100
120
140
(15)


De-Localized eigenvector- Defective mass
Amplitude of eigenvector components
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0
20
40
60
80
100
120
140
location
Participation
Ratio ≈ 85


Localized eigenvector- Defective mass
Amplitude of eigenvector components
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0
20
40
60
80
100
120
140
location
Participatio Ratoo ≅ 19

```
#!/usr/bin/env python
#  Note: Original program is courtesy of Gabe Herczeg
#  a student in 204A class


from __future__ import division
import numpy
from matplotlib.pylab import *

N = 128
M = numpy.zeros([N,N])
m = 1.0
k = 1.0
e = 0.1

M[0,N-1] = -k/(m*(1-e))
M[0,0]= 2*k/(m*(1-e))
M[0,1] = -k/(m*(1-e))

for i in range(1,N-1):
    M[i,i-1] = -k/m
    M[i,i] = 2*k/m
    M[i,i+1] = -k/m

M[N-1,0] = -k/m
M[N-1,N-1]= 2*k/m
M[N-1,N-2] = -k/m


eigvals,eigvectors = numpy.linalg.eig(M)
print "\n Defective mass at l =0 location case"
print "\n Calculated Eigenvalues from matrix"
print (sort(eigvals))
##theory_eigval = []
##for n in range(1, N+1):
##    q = (2*numpy.pi*n)/N
##    theory_eigval += [2 * (1 - math.cos(q))]
##print "\n Theoretical Eigenvalues:"
##print (sort(theory_eigval))

print "\n Theoretical frq**2"
y = 4/(1-e**2)
print y

pr = numpy.zeros(N)
for i in range(N):
    pr[i] = 1.0/(numpy.sum(abs(eigvectors[i])**4))

plot(range(N),pr,'g^')
title('Participation Ratio -Defective mass')
ylabel('Participation Ratio')
xlabel('location')
grid()
savefig('defect_mass_pr.pdf')
clf()

print "\n Localized eigenvector & participation ratio"
x = 0
print x, pr[x]
clf()
plot(abs(eigvectors[x]),'g^')
title('Localized eigenvector- Defective mass')
ylabel('Amplitude of eigenvector components')
```

```
xlabel('location')
grid()
savefig('amp_of_loc_eigvect.pdf')

print "\n De-Localized eigenvector & participation ratio"
x = 80
print x, pr[x]
clf()
plot(abs(eigvectors[x]),'g^')
title('De-Localized eigenvector- Defective mass')
ylabel('Amplitude of eigenvector components')
xlabel('location')
grid()
savefig('amp_of_deloc_eigvect.pdf')
```