PHYSICS 204A

Fall, 2010

HW # 2 SOLUTIONS

Physics 204A

HOMEWORK #2

1 $(AB)^\dagger_{ij} = (AB)^*_{ji}$ This is by definition:

The matrix element of the Hermitian conjugate of an operator are the complex conjugates of the elements of the transpose.

Now

$$(AB)_{ji} = \sum_k A_{jk} B_{ki}$$ by def of matrix multiplication

so $$(AB)^*_{ji} = \sum_k A^*_{jk} B^*_{ki}$$ (properties of adding/multiplying complex numbers)

Finally $$A^*_{jk} = (A^\dagger)_{kj}$$
$$B^*_{ki} = (B^\dagger)_{ik}$$ (def of Hermitian conjugation)

Putting this together

$$(AB)^\dagger_{ij} = \sum_k (A^\dagger)_{kj} (B^\dagger)_{ik}$$
$$= \sum_k (B^\dagger)_{ik} (A^\dagger)_{kj} = (B^\dagger A^\dagger)_{ij}$$

The last step is again the definition of matrix multiplication.

Since all components of $(AB)^\dagger$ equal those of $B^\dagger A^\dagger$ the operators themselves must be equal.

2 $C_i = (A \times B)_i = \epsilon_{ijk} A_j B_k$

(a)

check it out $\quad C_1 = \epsilon_{1jk} A_j B_k = \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2$

$$C_1 = A_2 B_3 - A_3 B_2 \qquad \checkmark\checkmark$$

$$C_2 = \epsilon_{2jk} A_j B_k = \epsilon_{213} A_1 B_3 + \epsilon_{231} A_3 B_1$$

$$= -A_1 B_3 + A_3 B_1 \qquad \checkmark\checkmark$$

(b)

$$\vec{A} \cdot (\vec{A} \times \vec{B}) = A_i (\vec{A} \times \vec{B})_i$$

$$= A_i \epsilon_{ijk} A_j B_k = \underbrace{\epsilon_{ijk}}_{\text{anti-symmetric}} \underbrace{A_i A_j}_{\text{symmetric}} B_k$$

$$\therefore \; = 0$$

property of ϵ_{ijk}

More slowly $\quad \epsilon_{ijk} A_i A_j B_k = -\epsilon_{jik} A_i A_j B_k$

$$= -\epsilon_{ijk} A_j A_i B_k \qquad \text{(interchanging names of dummies } i, j\text{)}$$

$$= -\epsilon_{ijk} A_i A_j B_k$$

since A_i are #'s which commute

Since $\boxtimes = -\boxtimes$ it must vanish.

2c $\quad \left[A\times(B\times C)\right]_i = \epsilon_{ijk} A_j (B\times C)_k$ (sums on j, k understood)

$$= \epsilon_{ijk} A_j \epsilon_{k\ell m} B_\ell C_m$$

We need the following identity

(sum on k understood) $\longrightarrow$ $\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$

If we assume this is true

$$\left[A\times(B\times C)\right]_i = A_j B_\ell C_m (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})$$

$$= A_m B_i C_m - C_i A_j B_j$$

$$= (\vec{A}\cdot\vec{C})\, B_i - (\vec{A}\cdot\vec{B})\, C_i$$

Since this is true for all components i

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\,\vec{B} - (\vec{A}\cdot\vec{B})\,\vec{C}$$

proof of identity: $\epsilon_{ijk} = 0$ unless i, j, k all different so given k the pair i, j has only two choices for a nonzero result. Since the same k appears in $\epsilon_{k\ell m}$ the pair ℓ, m can also have only the same two values. The only thing to check is whether the order is the same or reversed. If the same we have a perfect square and get 1. If reversed, we can exchange but pick up a $-$ sign.

3. The matrix is block diagonal so consider the 2x2 blocks independently.

The lower block cannot be diagonalized !! But fortunately it is still easy to exponentiate:

$$e^{\begin{pmatrix}0&1\\0&0\end{pmatrix}} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix} + \tfrac{1}{2}\underbrace{\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}}_{=\begin{pmatrix}0&0\\0&0\end{pmatrix}} + \dots = \begin{pmatrix}1&1\\0&1\end{pmatrix}$$

so all higher terms vanish also

To deal with upper block, diagonalize it

$$\begin{vmatrix}-\lambda & 1\\ 1 & -\lambda\end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

$$\lambda = 1 \quad |e_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \lambda = -1 \quad |e_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

So transformation matrix $S = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$

Thus $$e^{\begin{pmatrix}0&1\\1&0\end{pmatrix}} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1&+1\\1&-1\end{pmatrix}}_{S^T}\underbrace{\begin{pmatrix}e^1&0\\0&e^{-1}\end{pmatrix}}_{e^D}\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}}_{S}$$

$$= \frac{1}{2}\begin{pmatrix}e & +1/e\\ e & -1/e\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}e+1/e & e-1/e\\ e-1/e & e+1/e\end{pmatrix} = \begin{pmatrix}\cosh 1 & \sinh 1\\ \sinh 1 & \cosh 1\end{pmatrix}$$

So whole answer is $$\exp\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix} = \begin{pmatrix}\cosh 1 & \sinh 1 & 0 & 0\\ \sinh 1 & \cosh 1 & 0 & 0\\ 0&0&1&1\\0&0&0&1\end{pmatrix}$$

NB You can also do this upper block by noting $\begin{pmatrix}0&1\\1&0\end{pmatrix}^2 = \begin{pmatrix}1&0\\0&1\end{pmatrix} = I$
This is the trick usually used in QM texts when exponentiating spin 1/2 Pauli matrices.

4 First note it is true for 1×1 matrices, aka real numbers. (It is always a good idea to check matrix identities in this 1 dimensional limit!)

$$A = a$$

dimension 1 a number

$$\frac{\partial}{\partial a} \ln(\det a) = \frac{\partial}{\partial a} \ln a = 1/a = A^{-1}$$

det of 1×1 matrix = the only matrix element

inverting a 1×1 matrix...

Use the standard definition of derivative $\frac{f(a+\epsilon)-f(a)}{\epsilon}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & \\ a_{21} & a_{22} & & \\ & & a_{ij}+\epsilon & \\ & & & \end{bmatrix}$$

By linearity of determinant the determinant with $a_{ij} \to a_{ij}+\epsilon$ is the original determinant $+\,\epsilon\, C_{ij}$ ← the cofactor.

4(cont'd) But we know $[A]^{-1}_{ji} = \dfrac{C_{ij}}{\det A}$

So $$\frac{\partial}{\partial a_{ij}} \ln(\det A) = \frac{1}{\epsilon}\left[\ln(\det A + \epsilon C_{ij}) - \ln(\det A)\right]$$

$$= \frac{1}{\epsilon}\left[\ln \det A\left(1 + \epsilon \frac{C_{ij}}{\det A}\right) - \ln \det A\right]$$

$$= \frac{1}{\epsilon}\left[\ln(\det A)\, \ln(1 + \epsilon A^{-1}_{ji}) - \ln \det A\right]$$

$$= \frac{1}{\epsilon}\left[\ln(\det A)\left[1 + \epsilon A^{-1}_{ji}\right] - \ln \det A\right]$$

$$= A^{-1}_{ji}$$

An alternate proof can be constructed via QM perturbation theory

Alternate "QM-style" argument: Consider
$$A + \epsilon \begin{pmatrix} \emptyset & & \emptyset \\ \emptyset & 1 & \emptyset \end{pmatrix} \leftarrow \text{row } i$$
$$\uparrow \text{col. } j$$
and think of second matrix as "perturbation", V.

$$\det(A+V) = \det A(I + A^{-1}V) = \det A \det(I + A^{-1}V)$$

$$[A^{-1}V]_{nm} = A^{-1}_{nk} V_{km} = A^{-1}_{nk}\epsilon \delta_{ik}\delta_{jm} = \epsilon A^{-1}_{ni}\delta_{jm}$$

Writing it out as a real matrix $A^{-1}V = \epsilon \begin{pmatrix} A^{-1}_{1i} \\ A^{-1}_{2i} \\ \vdots \end{pmatrix}$

$\uparrow$ jth col

The jth col. is filled by the ith col. of A^{-1}

Now $\det\left[I + A^{-1}V\right] = 1 + \epsilon A^{-1}_{ji}$

4 cont'd

Thus $\det(A+V) = (\det A)\left(1 + \epsilon A^{-1}_{ji}\right)$

The derivative is then, finally,

$$\frac{\det(A+V) - \det A}{\epsilon} = A^{-1}_{ji}$$

5. Our original matrix for A is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(in the bases $|e_1\rangle$, $|e_2\rangle$)

You can construct the new matrix for A using the procedure of sandwiching A between S and S^T where S describes the change in bases from $|e_i\rangle$ to $|f_j\rangle$. But that involves recalling where to put the S and S^T.

Maybe it is less demanding on the memory simply to say

$$(A')_{11} = \langle f_1|\hat{A}|f_1\rangle = \frac{1}{\sqrt{2}}(1\ 1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ (working in $|e_i\rangle$ bases)

$$= \frac{1}{2}(1\ 1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

$$(A')_{12} = \langle f_1|\hat{A}|f_2\rangle = \frac{1}{2}(1\ 1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$$

$$(A')_{21} = 1 \quad \text{(by symmetry)}$$

$$(A')_{22} = \langle f_2|\hat{A}|f_2\rangle = \frac{1}{2}(1\ -1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

So $\hat{A}$ in new bases $\{|f_i\rangle\}$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

6 Physical results do not depend on bases choice. First, the possible results of measurements are eigenvalues and they are indep of bases

$\boxed{6}$ cont'd **Proof** Suppose λ obeys $\det(A-\lambda I)=0$

Then because $\det ABC = \det A \det B \det C$ and because the procedure for changing bases involves a unitary transformation $A' = UAU^+$ with $UU^+ = 1$ we see

$$\det(A-\lambda I) = \det U(A-\lambda I)U^+ = \det(A'-\lambda I)$$

so λ also solves $\det(A'-\lambda I)=0$. In other words λ is also an eigenvalue of A'.

Furthermore any expectation value is basis independent

$$\langle\psi'|A'|\psi'\rangle = \langle\psi|U^+(UAU^+)U|\psi\rangle = \langle\psi|A|\psi\rangle$$

For problem 5 Note original eigenvalues of $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ are $\lambda=\pm1$ as are eigenvalues of $\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

If the state $|\psi\rangle = \begin{pmatrix}a\\b\end{pmatrix}$ in $|e_i\rangle$ basis ~~to~~

$$\langle\psi|A|\psi\rangle = (a^* \; b^*)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = |a|^2-|b|^2$$

Meanwhile $a|e_1\rangle + b|e_2\rangle = a\frac{1}{\sqrt2}(|f_1\rangle+|f_2\rangle) + b\frac{1}{\sqrt2}(|f_1\rangle - |f_2\rangle)$

$$= \frac{1}{\sqrt2}(a+b)|f_1\rangle + \frac{1}{\sqrt2}(a-b)|f_2\rangle$$

$$\langle\psi'|A'|\psi'\rangle = \frac12 (a^*+b^* \;\; a^*-b^*)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a+b\\a-b\end{pmatrix} = \frac12(a^*+b^* \;\; a^*-b^*)\begin{pmatrix}a-b\\a+b\end{pmatrix}$$

$$= \frac12\left(|a|^2-|b|^2-a^*b+b^*a+|a|^2-|b|^2-b^*a+a^*b\right)$$

$$= |a|^2-|b|^2$$

```
#include <stdio.h>
#include <math.h>

int main(void)

{

int i,j;
float M[10][10],V[10],W[10];
FILE * fileout;

fileout=fopen("hermione","w");

for (i=0; i<10; i=i+1)
{
V[i]=0.2*i-0.01*i*i;
}

for (i=0; i<10; i=i+1)
{
for (j=0; j<10; j=j+1)
{

M[i][j] = 0.5 + 20.0 / ( (i+1)  + 3.0 * (j+1) ) ;

}
}

for (i=0; i<10; i=i+1)
{
W[i]=0.0;
for (j=0; j<10; j=j+1)
{

W[i]=W[i] + M[i][j]*V[j];

}
}

for (i=0; i<10; i=i+1)
{
fprintf(fileout,"\n%d  %12.6lf   %12.6lf",i,V[i],W[i]);
}

return 0;
```

$W(7) \cong 9.566....$

```
}
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