PHYSICS 204A

Fall, 2010

HW #1 SOLUTIONS

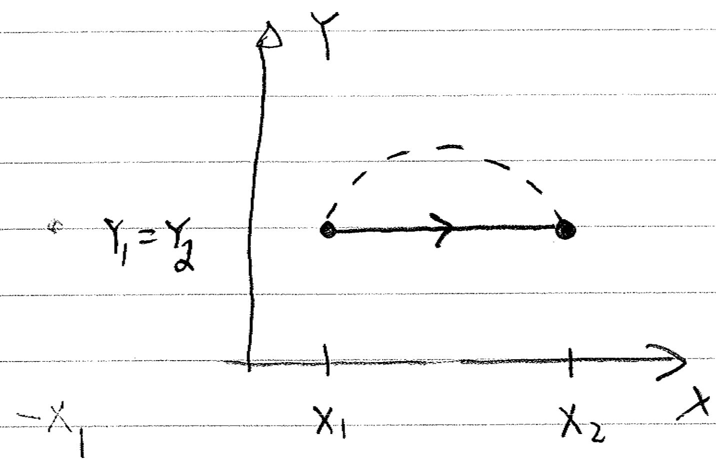
Problem 1 : GOLDBART - STONE 1.3

Hyperbolic Geometry

(The non Euclidean Geometry of Lobachevsky)

Metric $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$ $(y > 0)$

physically this means "distance traveled" is less as one moves to higher y values.



instead of ——
should go to higher y
to minimize "distance" - - -

$$\int_{x_1}^{x_2} \frac{(1+y'^2)^{1/2}}{y}\, dx$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = -\frac{(1+y'^2)^{1/2}}{y^2} - \frac{d}{dx}\left(\frac{(1+y'^2)^{-1/2}\frac{1}{2}2y'}{y}\right)$$

Could work through algebra, but we know general principle
since f is indep of x $\longrightarrow$ $f - y'\dfrac{\partial f}{\partial y'} = c$

$$\frac{(1+y'^2)^{1/2}}{y} - y'\,\frac{(1+y'^2)^{-1/2}\,y'}{y} = \frac{1}{R}$$ call constant $\frac{1}{R}$

Multiply by $y\sqrt{1+y'^2}$ yields

$$1 + y'^2 - y'^2 = \frac{1}{R} y \sqrt{1+y'^2}$$

Thus $\frac{R}{y} = \sqrt{1+y'^2}$ $\quad \frac{R^2}{y^2} - 1 = y'^2$ $\quad \sqrt{\frac{R^2-y^2}{y^2}} = \frac{dy}{dx}$

Finally $\frac{y\,dy}{\sqrt{R^2-y^2}} = dx$ or $-(R^2-y^2)^{1/2} = x + a$

So $R^2 - y^2 = (x+a)^2$ $\quad \boxed{(x+a)^2 + y^2 = R^2}$

Circle center at $(-a, 0)$ radius R,

Given points (x_1, y_1) $(x_2\, y_2)$ how are a, R determined

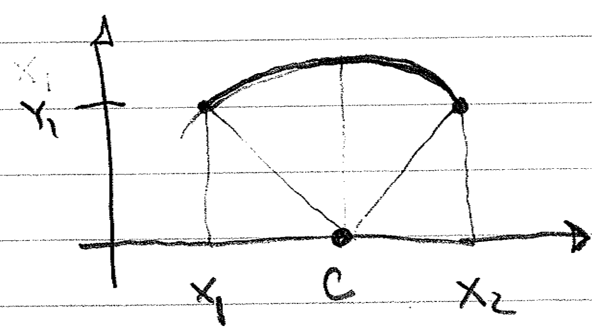
$$\begin{aligned}(x_1+a)^2 + y_1^2 &= R^2 \\ (x_2+a)^2 + y_2^2 &= R^2\end{aligned} \implies (x_1+a)^2 + y_1^2 = (x_2+a)^2 + y_2^2$$

to get a

$$x_1^2 + 2ax_1 + \cancel{a^2} + y_1^2 = x_2^2 + 2ax_2 + \cancel{a^2} + y_2^2$$

$$a = \frac{x_1^2 - x_2^2 + y_1^2 - y_2^2}{2(x_2 - x_1)}$$

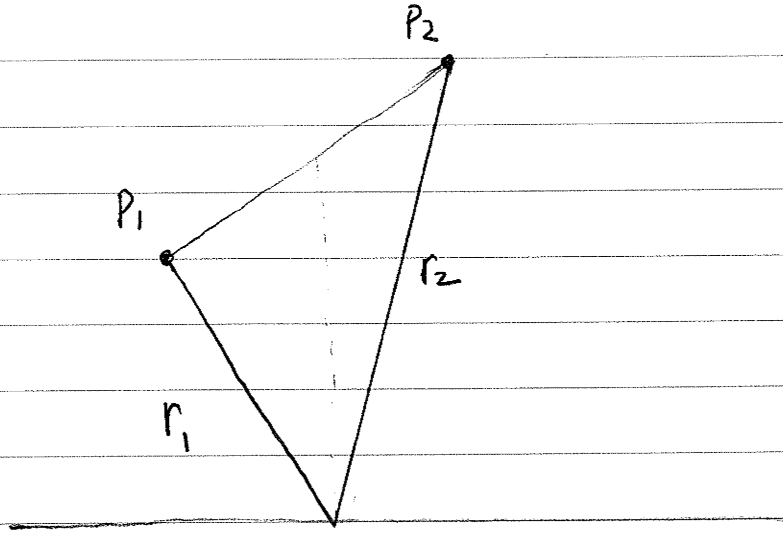
Special case $y_1 = y_2$ $\quad -a = \left(\frac{x_1+x_2}{2}\right)$ as expected



$$R^2 = y_1^2 + \left(\frac{x_1 - x_2}{2}\right)^2$$

(Pythagorean thm)

Geometrical construction :



using $-a = \frac{x_1 + x_2}{2}$

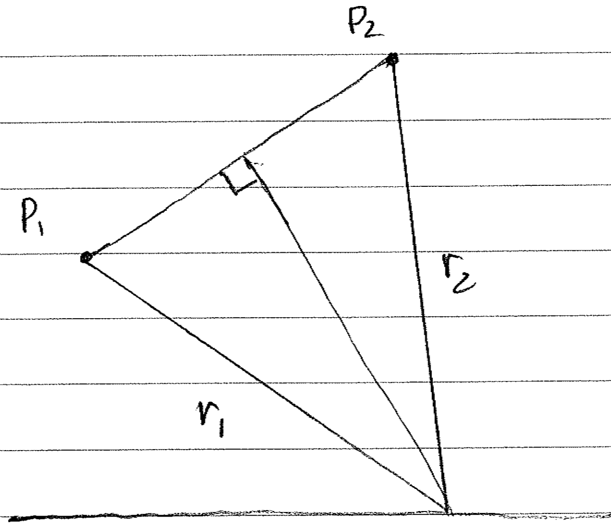
evidently will fail since

$r_1 \neq r_2$

Must move to right.

$\boxed{Q}$ pretty clearly $r_1 = r_2$ when using $\perp$ bisector

by "SAS"



Page CV13B shows

mathematically that $(-a, 0)$

is indeed on $\perp$ bisector.

Recast Eqn in clever way

$$\frac{y_2^2 - Y_1^2}{2} = \frac{X_1^2 - X_2^2}{2} + a(x_1 - x_2)$$

$$\left(\frac{y_1 + y_2}{2}\right)(y_2 - y_1) = (x_1 - x_2)\left[\frac{x_1 + x_2}{2} + a\right]$$

$$-\frac{y_1 + y_2}{2} = \frac{x_1 - x_2}{y_2 - y_1}\left[-a - \frac{x_1 + x_2}{2}\right]$$

Think about this as eqn of line

$$y - y_0 = m(x - x_0)$$

through point $(-a, 0) = (x, y)$

Evidently $y_0 = \frac{Y_1 + y_2}{2}$ $X_0 = \frac{X_1 + X_2}{2}$

What is this point? Midpoint P of (X_1, X_2) (Y_1, Y_2) !

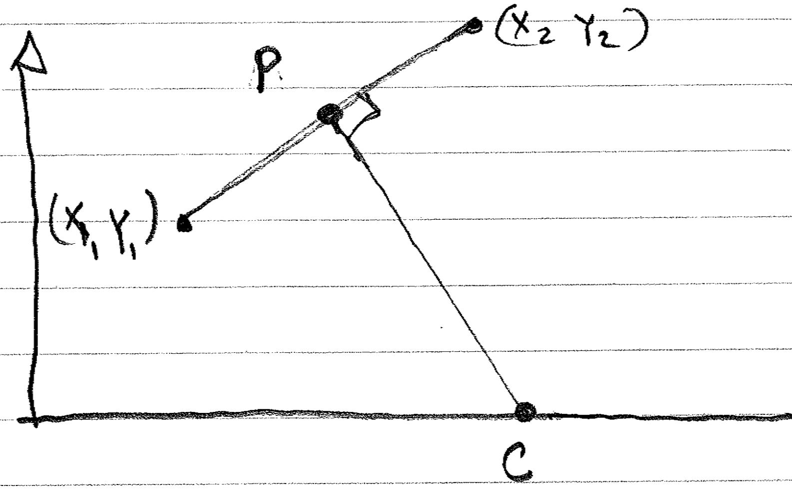
slope of line is $\frac{X_1 - X_2}{Y_2 - Y_1}$ Does any one recognize this?

It is $-1/m$

where m is slope

of line through P_1, P_2

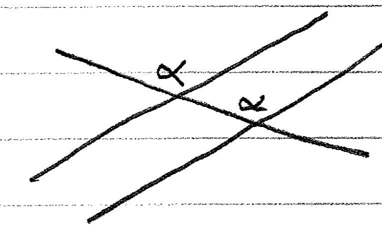
→ perpendicular



Before doing parts b) c) some preliminary comments:

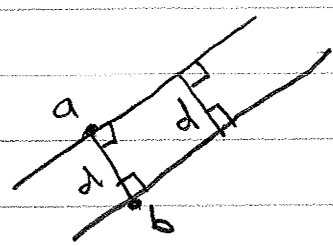
1) In Euclidean geometry there are many equivalent definitions of the word "parallel".

a) 2 lines are parallel if a third line intersects them at the same angle



b) two lines are parallel if they never meet

c) two lines are parallel if the distance between them is constant.



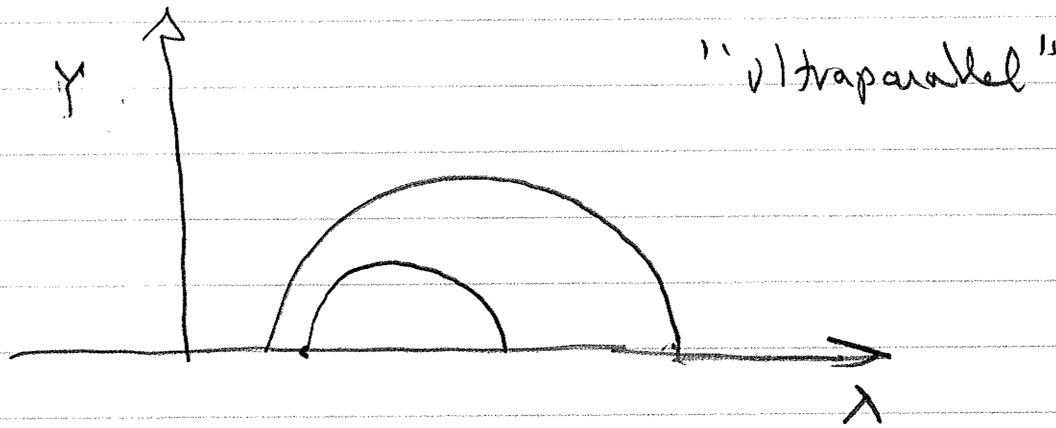
(This picture raises question about how points a/b chosen)

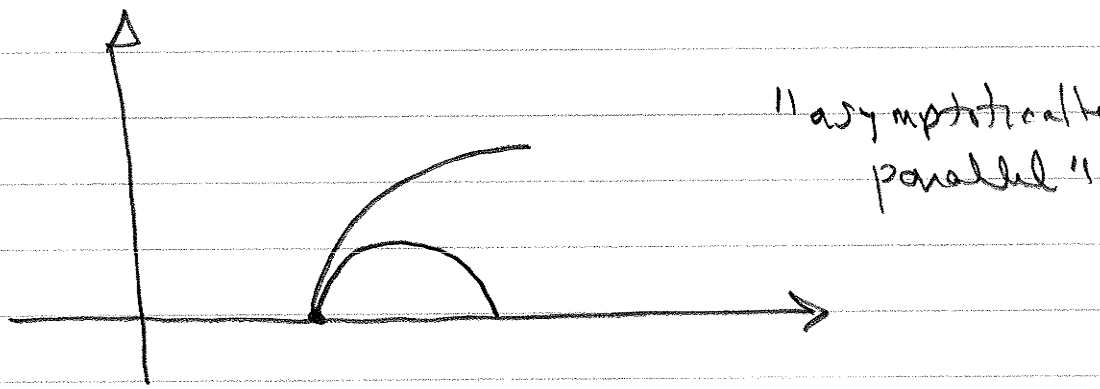
In non Euclidean geometries these definitions are inequivalent!

2) So the textbook adopts the definition that lines are parallel if they meet at infinity at one end, that is definition to do the problem as stated. The definition sort of makes sense in that at one end at least it is a version of c. Clearly 2 lines in this geometry must touch on the x axis or else the distance between them is infinite. (It only sort of works because of presence of non touching end)

3) This is apparently <u>not</u> the usual nomenclature.

Carlip says word "parallel" is not usually used in this geometry.





b) Uniqueness follows from algebra/geometric construction:

There is only one way to do it!

c) $(X+a)^2 + y^2 = R^2$

$$y = \left[R^2 - (x+a)^2\right]^{1/2}$$

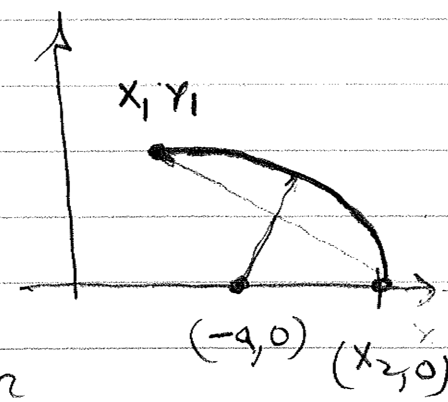
$$dy/dx = \left[R^2 - (x+a)^2\right]^{-1/2}\left[-(x-a)\right]$$

$$(dy/dx)^2 = (X+a)^2 / R^2 - (X+a)^2$$

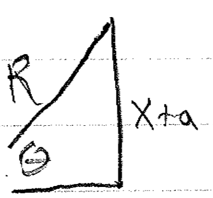
$$1 + (dy/dx)^2 = R^2 / R^2 - (x+a)^2$$

$$\therefore \int_{X_1}^{X_2} \frac{dx\left[1+(dy/dx)^2\right]^{1/2}}{y}$$

$$= \int_{X_1}^{X_2} dx \frac{R}{R^2 - (x+a)^2}$$



Note that $y_2 = 0 \Rightarrow (X_2 + a)^2 = R^2$



$(X+a) = R\sin\theta \qquad dx = R\cos\theta\, d\theta$

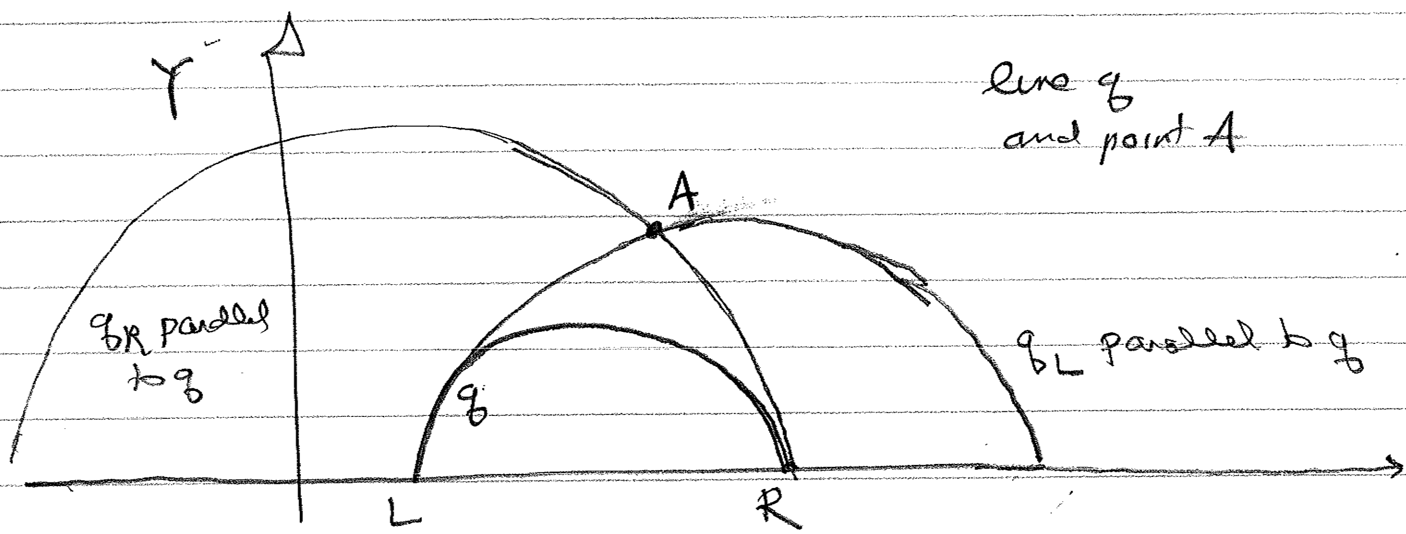
$$\int \frac{R\cos\theta\, d\theta}{R^2\cos^2\theta} R = \int \sec\theta\, d\theta = \ln(\sec\theta + \tan\theta)$$

Travel time from (x_1, y_1) to $(x_2 \cdot 0)$ is
↙ nonzero

$$\ln(\sec\theta + \tan\theta) = \ln\left[\frac{R + (x+a)}{\sqrt{R^2 - (x+a)^2}}\right]_{x_1}^{x_2}$$

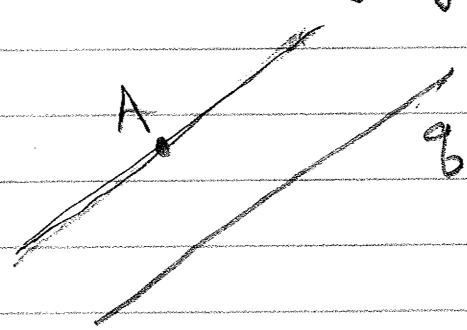
But $R^2 - (x_2 + a)^2 = 0$ so time diverges ∎

Lines go from $+\infty$ to $-\infty$: Start and end on x-axis



two ways ~~to~~ meet at infinity : left or Right sides ⟹ 2 lines.

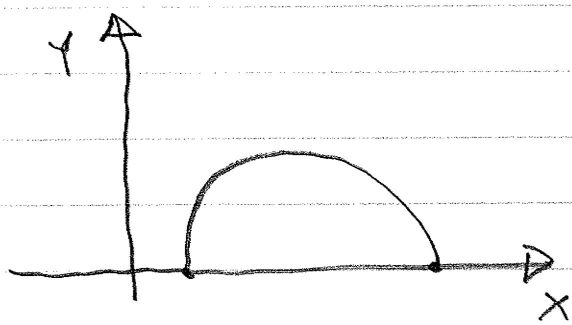
What is point : In "ordinary" geometry only 1 line through A which never meets q



Q: What is a line??

A: Can travel along it forever without reaching end.

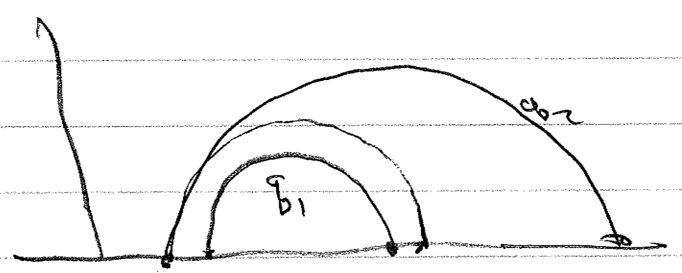
If so, in this geometry, lines begin and end on x-axis



"Usual" geometry: only one line through 2 given points. Same is true here.

"Usual" geometry:

Basic Q: why is meeting at ∞ part of definition of parallel. Why not just that they do not meet?!



why are q_1 q_2 not parallel?!

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Problem Set #1

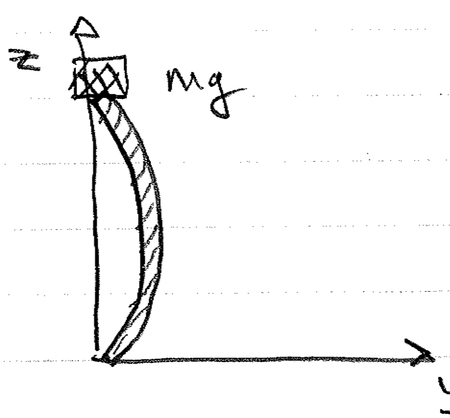
Problem 2

$\boxed{\text{Goldbart - Stone 1.4}}$

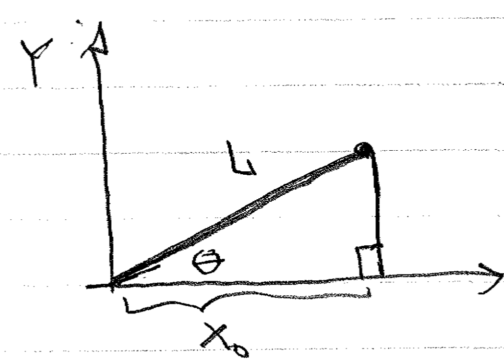
a) We will first just use the formula for the elastic energy (later we will see where it comes from!!) of the bar

$$U_e = \int_0^L \tfrac{1}{2}\, Y I\, y''\, dz$$

We need the potential energy of the load mg



How much lower does mg fall due to bending?



When a line segment length L makes an angle θ with an axis

its projection $X_0 = L\cos\theta = L(1-\theta^2/2)$ if θ is small

Now $\theta \doteq \tan\theta = dy/dx$ so $L - X_0 = -\frac{1}{2}(dy/dx)^2$

Considering a bent shape as a collection of such segments we see that the load mg falls a distance $\frac{1}{2}\int dz \left(\frac{dy}{dz}\right)^2$

Putting together

$$\mathcal{U}[y] = \int_0^L \left[\frac{YI}{2}(y'')^2 - \frac{Mg}{2}(y')^2\right] dz$$

$$\text{If } y(z) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi z}{L} \quad \leftarrow \text{ note } y(0) = y(L) = 0$$

$$y'(z) = \sum_{n=1}^{\infty} a_n \frac{n\pi}{L} \cos\frac{n\pi z}{L}$$

$$y''(z) = -\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi z}{L}$$

$$\int_0^L \cos^2\left(\frac{n\pi z}{L}\right) dz = L/2 = \int_0^L \sin^2\left(\frac{n\pi z}{L}\right) dz$$

$$\mathcal{U}(y) = \sum_{n=1}^{\infty} \left[\frac{YI}{2}\left(\frac{n\pi}{L}\right)^4 - \frac{Mg}{2}\left(\frac{n\pi}{L}\right)^2\right] a_n^2 \frac{L}{2}$$

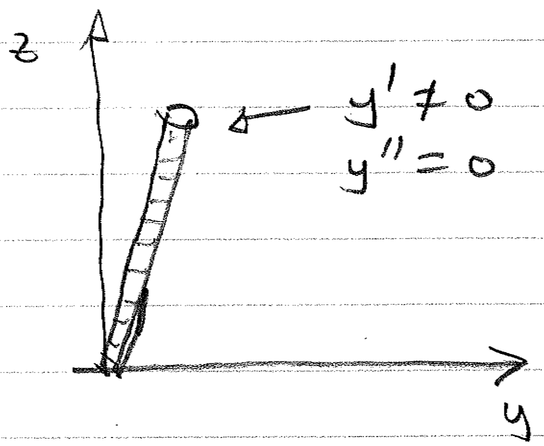
Unstable if $\frac{YI}{2}\left(\frac{n\pi}{L}\right)^2 < \frac{Mg}{2}$ $\boxed{Mg > YI\left(\frac{n\pi}{L}\right)^2}$

Physically reasonable M in numerator and L in denominator ??

Now where does $\mathcal{U}_e$ come from?

__First__ __Comment__ on $\mathcal{U}_e = \int_0^L \frac{1}{2} YI\, y''\, dz$:

Clearly if you do not bend the bar but just give it a nonzero y' it remains straight and the elastic energy is zero



Thus we expect

$\mathcal{U}_e \sim y''$

__not__ $\mathcal{U} \sim y'$!

Put better: The curvature of a function is ~~not~~ set by the derivative.

~~Consider a circle~~ $\cancel{x^2 + y^2 = R^2}$ $\cancel{y = (R^2 - x^2)^{1/2}}$

~~Then~~ $\cancel{dy/dx = -x(R^2 - x^2)^{-1/2}}$

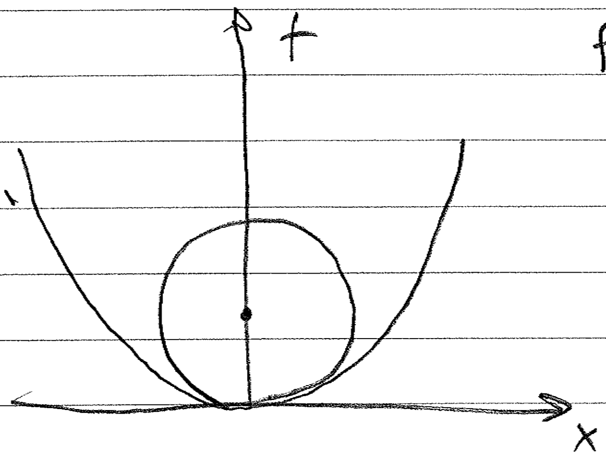
$\cancel{d^2y/dx^2 = -(R^2 - x^2)^{-1/2} - x^2 (R^2 - x^2)^{-3/2}}$

What circle best approximates

$$f(x) = f(x_0) + f'(x_0) + \tfrac{1}{2} f''(x_0)(x-x_0)^2 + \dots$$

$$x_0 = 0$$

wlog consider $f(x_0) = 0 = f'(x_0)$ i.e. orient axes and locate axes at minimum of $f(x)$



$$f(x) = \tfrac{1}{2} f''(0)\, x^2$$

$$x^2 + (y-R)^2 = R^2$$

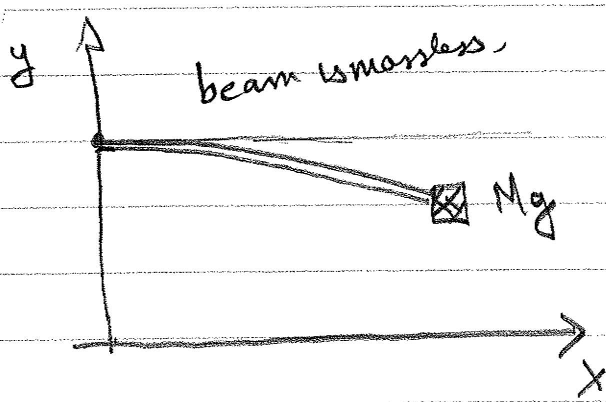
$$y = R - \sqrt{R^2 - x^2}$$

$$= R - R\left(1 - x^2/R^2\right)^{1/2}$$

Taylor expand $\quad y = R - R\left(1 - x^2/2R^2\right) = \frac{1}{2}\frac{1}{R}x^2$

Matching up we see $\quad f''(0) = 1/R$

(b) We use $U_e = \int_0^L \frac{YI}{2}(y'')^2 dx$



$$U = \int_0^L \frac{YI}{2}(y'')^2 + Mg\,y(L)$$

(Use x as horiz. axis not z as stated in text)

Question: Why not use $\int_0^L y'\,dx$ as distance Mg falls?

<u>Answer</u>: This is same as $y(L)$!!

So, our total energy functional is:

$$U[y] = \int_0^L \frac{YI}{2}(y'')^2 dx + Mg\,y(L)$$

Applying integration by parts twice we get:

$$\delta U = \int_0^L YI\,y''\,\delta y''\,dx + Mg\,\delta y(L)$$

$$= YI\,y''\,\delta y'\Big|_0^L - \int_0^L YI\,\delta y'\,y'''\,dx + Mg\,\delta y(L)$$

$$\delta U = \underset{1}{YI\,y''\,\delta y'\Big|_0^L} - \underset{2}{YI\,y'''\,\delta y\Big|_0^L} + \underset{3}{\int_0^L YI\,y^{4}\,\delta y\,dx} + \underset{4}{Mg\,\delta y(L)}$$

<u>Note</u>: we could not naively apply E-L equations since endpoints are <u>not</u> fixed, but we are given specific boundary conditions which help us solve the problem.

We are given boundary conditions:

At $x = 0$, $y(0) = 0$ and $y'(0) = 0$

which means $y'''(0) = y^{4}(0) = 0$ also

Re-writing δU (and eliminating zero terms based on above)

$$\delta U = YI y''(L)\, \delta y' - YI y'''(L)\, \delta y(L) + Mg\, \delta y(L) + \int_0^L YI y^{(4)} \delta y \, dx$$

$$= \boxed{YI y''(L)}\, \delta y' - \boxed{\left[YI y'''(L) - Mg \right]}\, \delta y(L) + \boxed{\int_0^L YI y^{(4)} dx}\, \delta y$$

All terms in boxes must vanish at a minimum.

So, we obtain:

$$y''(L) = 0$$
$$y'''(L) = MG / YI$$

From here, we let:

$y(x) = ax^3 + bx^2 + cx + d$ [$y^{(4)} = 0$ above!]

and solve for coefficients:

$c, d = 0$ since $y(0) = d$ and $y'(0) = c$

So $y(x) = ax^3 + bx^2$.

$$y''(x) = 6ax + 2b \Rightarrow y''(L) = 6aL + 2b = 0$$

$$\boxed{b = -3aL}$$

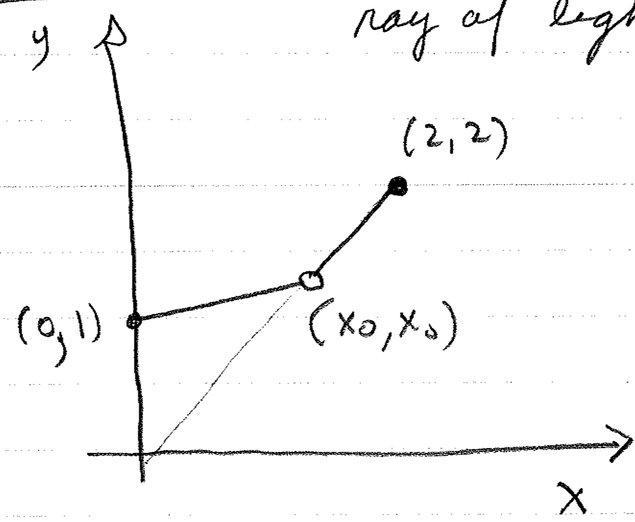
$$y'''(x) = 6a = Mg/YI \qquad a = Mg/6YI$$

Finally,

$$\boxed{y(x) = \underbrace{\frac{Mg}{6YI}}_{a} x^3 - \underbrace{\frac{3Mg}{6YI}}_{b} x^2 = \frac{Mg}{6YI}\left(x^3 - 3Lx^2\right)}$$

and $\boxed{y(L) = -\frac{1}{3} MgL^3/YI}$

Problem 3: minimize time travel of ray of light



$$n = a + b(y^2 - x^2)$$

$$v = c/n$$

$$dt = ds/v = ds\, n/c$$

$$y = mx + 1$$

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1+m^2} \quad \text{where } m = \frac{x_0 - 1}{x_0}$$

$$t_1(x_0) = \int_0^{x_0} dx\sqrt{1+m^2}\ \frac{a + b[(mx+1)^2 - x^2]}{c}$$

$$t_2(x_0) = (2 - x_0)\sqrt{2}\ \frac{a}{c}$$

$$t_1(x_0) = \frac{\sqrt{1+m^2}}{c}\int_0^{x_0} dx\ \ a + b[(m^2-1)x^2 + 2mx + 1]$$

$$= \frac{\sqrt{1+m^2}}{c}\ \ ax + b(m^2-1)\frac{x^3}{3} + mbx^2 + bx\ \Big|_0^{x_0}$$

$$= \frac{\sqrt{1+m^2}}{c}\left[(a+b)x_0 + mbx_0^2 + \frac{b(m^2-1)}{3}x_0^3\right]$$

$$\frac{dt}{dx_0} = \frac{d}{dx_0}(t_1 + t_2)$$ ← Note m is a function of x_0 here

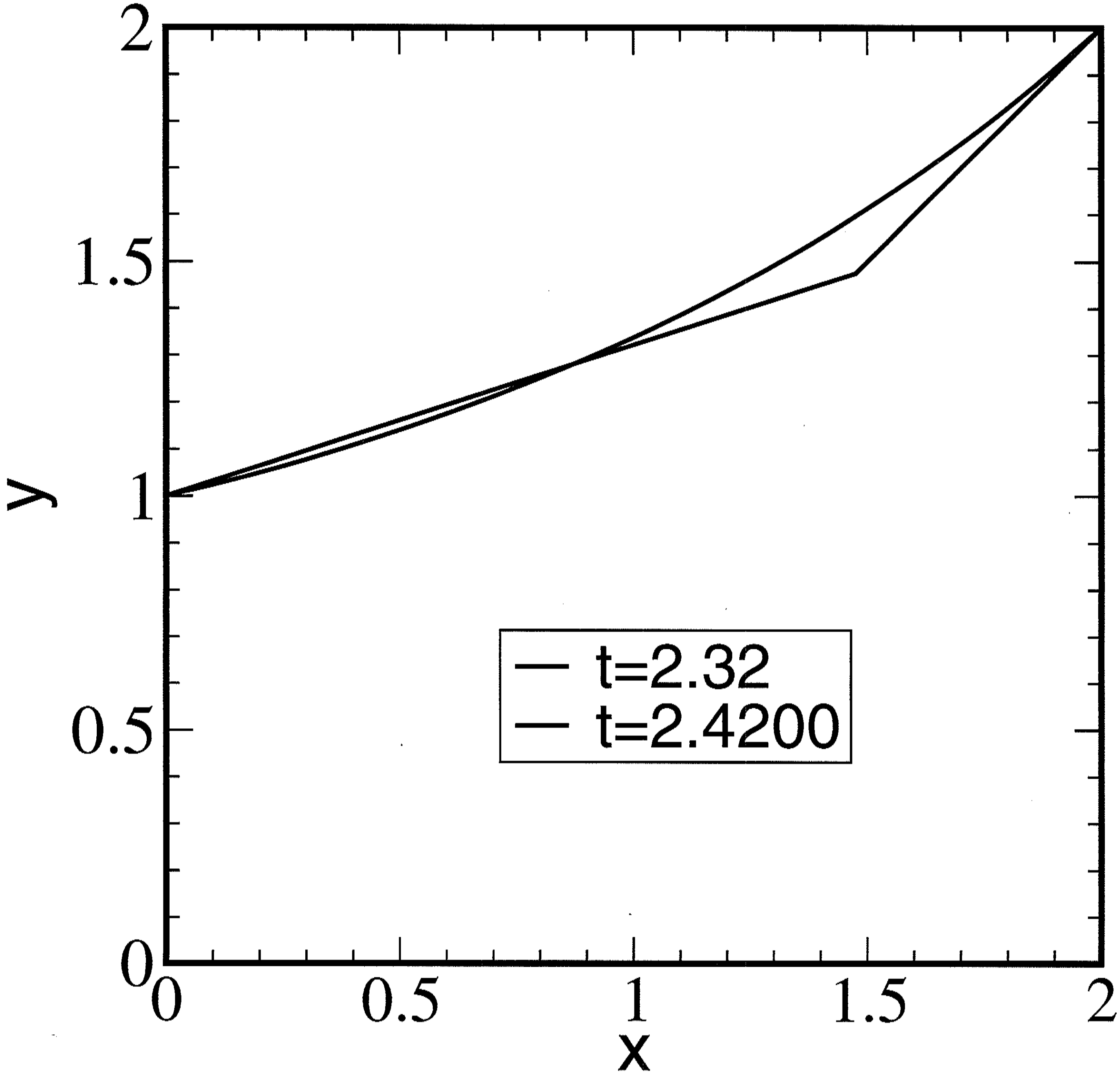
Program to compute minimum time: (3-2)


```
c          CODE TO COMPUTE MIN TIME TO TRAVEL FROM (1,0) to (2,2)
c          ALONG GENERAL PATH
c          FOR v=c/n  WITH c=1 and n=a+b |y^2-x^2|

           implicit none
           real*8 a,b,c,x(0:100),y(0:100),dx,dx2,n,beta,E,dE,move
           real*8 dy,ran2,xx,yy,avey(0:100),aveE,aveE2
           integer ix,Nx,imonte,Nmonte,iran,accept

           c=1.d0
           Nx=400

           write (6,*) 'enter a,b,Nmonte,move,beta,iran'
           read  (5,*)          a,b,Nmonte,move,beta,iran

           dx=2.d0/dfloat(Nx)
           dx2=dx*dx

c          INITIALIZE x,y TO BE STRAIGHT LINE

           do 100 ix=0,Nx
               x(ix)=0.d0+(2.d0-0.d0)*dfloat(ix)/dfloat(Nx)
               y(ix)=1.d0+(2.d0-1.d0)*dfloat(ix)/dfloat(Nx)
100        continue

c          COMPUTE "ENERGY"

           E=0.d0
           do 110 ix=1,Nx
               xx=x(ix)-dx2
               yy=(y(ix)+y(ix-1))/2.d0
               n=a+b*dabs(yy*yy - xx*xx)
               dE = dsqrt(  (y(ix)-y(ix-1))**2 + dx2 ) * n/c
               E=E+dE
               avey(ix)=0.d0
110        continue
           avey( 0)=1.d0*dfloat(Nmonte)
           avey(Nx)=2.d0*dfloat(Nmonte)

           accept=0
           aveE =0.d0
           aveE2=0.d0
           do 300 imonte=1,Nmonte
           do 200 ix=1,Nx-1

               xx=x(ix)-dx2

               yy=(y(ix)+y(ix-1))/2.d0
               n=a+b*dabs(yy*yy - xx*xx)
               dE =    - dsqrt(  (y(ix)-y(ix-1))**2 + dx2 ) * n/c

               yy=(y(ix)+y(ix+1))/2.d0
               n=a+b*dabs(yy*yy - xx*xx)
               dE = dE - dsqrt(  (y(ix)-y(ix+1))**2 + dx2 ) * n/c

               dy=move*(ran2(iran)-0.5d0)

               yy=(y(ix)+dy+y(ix-1))/2.d0
               n=a+b*dabs(yy*yy - xx*xx)
               dE =  dE + dsqrt(  (y(ix)+dy-y(ix-1))**2 + dx2 ) * n/c

               yy=(y(ix)+dy+y(ix+1))/2.d0
               n=a+b*dabs(yy*yy - xx*xx)
               dE =  dE + dsqrt(  (y(ix)+dy-y(ix+1))**2 + dx2 ) * n/c
```

3-3


```
                if (ran2(iran).le.dexp(-beta*dE)) then
                    accept=accept+1
                    y(ix)=y(ix)+dy
                    E=E+dE
                endif
                avey(ix)=avey(ix)+y(ix)
                aveE =aveE +E
c                aveE2=aveE2+E*E
200         continue
300         continue
            aveE =aveE /(dfloat(Nx-1)*dfloat(Nmonte))
c            aveE2=aveE2/(dfloat(Nx-1)*dfloat(Nmonte))

c         NEED TO BIN TO GET ACCURATE ERROR ESTIMATE...

          write (6,990) dfloat(accept)/(dfloat(Nx-1)*dfloat(Nmonte)),
c     1       aveE,dsqrt(aveE2-aveE*aveE)/dsqrt(dfloat(Nmonte-1))
     1      E,aveE
c990       format('acc= ',f8.4,' <t>= ',f8.4,' +- ',f8.4)
990       format('acc= ',f8.4,' <t>= ',f8.4,'     t= ',f8.4)
991       format(i6,3f8.4)

            do 400 ix=0,Nx
                  write (66,991) ix,x(ix),y(ix),avey(ix)/dfloat(Nmonte)
400         continue

            end


c         USE THESE COMMENTED OUT LINES IF REAL*8 DESIRED.
      REAL*8 FUNCTION RAN2(IDUM)
      IMPLICIT REAL*8(A-H,O-Z)
c      FUNCTION RAN2(IDUM)
      save
      PARAMETER (M=714025,IA=1366,IC=150889,RM=1.4005112E-6)
      DIMENSION IR(97)
      DATA IFF /0/
      IF(IDUM.LT.0.OR.IFF.EQ.0)THEN
        IFF=1
        IDUM=MOD(IC-IDUM,M)
        DO 11 J=1,97
          IDUM=MOD(IA*IDUM+IC,M)
          IR(J)=IDUM
11      CONTINUE
        IDUM=MOD(IA*IDUM+IC,M)
        IY=IDUM
      ENDIF
      J=1+(97*IY)/M
      IF(J.GT.97.OR.J.LT.1)PAUSE
      IY=IR(J)
      RAN2=IY*RM
      IDUM=MOD(IA*IDUM+IC,M)
      IR(J)=IDUM
      RETURN
      END
```

N_x=400   dy=0.01   β=10000



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