F-1

We know how $(ct, \vec{x})$ and $(\frac{E}{c}, \vec{p})$

transform from frame to frame

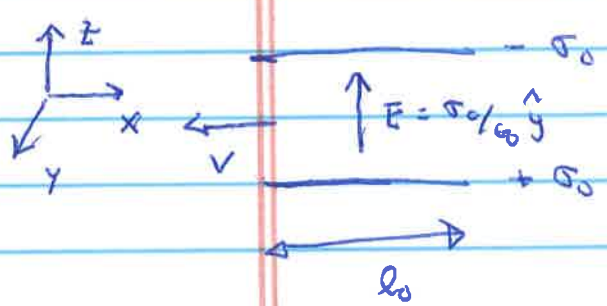
$$(a^\mu)' = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu a^\nu$$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What about $\vec{E}$ and $\vec{B}$?

Consider the most simple case $\vec{E}$ = uniform and $\vec{B} = 0$

as produced by 2 parallel plates of charge (capacitor)



In a frame moving in $\hat{x}$ direction the plates are

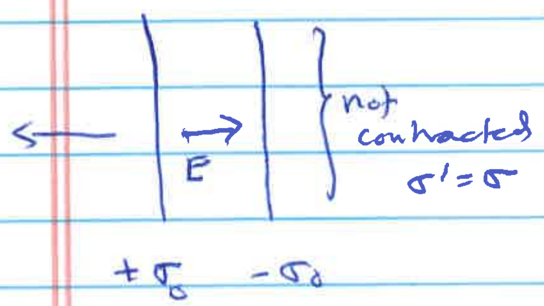
Lorentz contracted $\sigma' = \gamma \sigma_0 > \sigma_0$ $\gamma = 1/\sqrt{1 - v^2/c^2}$

$$E' = \gamma E$$

F-2

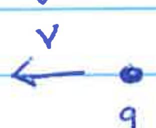
Comments (1) $\vec{E}'$ is not tilted away from $\hat{y}'$ axis by symmetry.

(2) only part of $\vec{E}$ which is $\perp$ to $\vec{v}$ is increased because only lengths parallel to $\vec{v}$ are contracted



makes q move to right

motion of frame S'



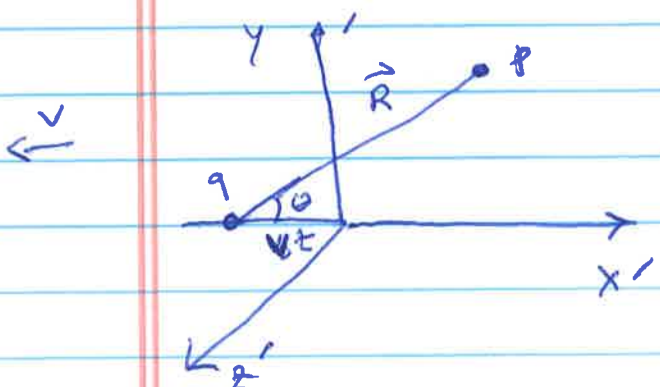
Apply to field of moving point charge

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} \qquad \begin{cases} E_x = \frac{q}{4\pi\epsilon_0}\frac{x}{r^3} \\ E_y = \frac{q}{4\pi\epsilon_0}\frac{y}{r^3} \\ E_z = \frac{q}{4\pi\epsilon_0}\frac{z}{r^3} \end{cases}$$

E_y and E_z are $\perp$ to $v\hat{x}$

$$E'_x = E_x \qquad x = \gamma(x' + vt)$$
$$E'_y = \gamma E_y \qquad y = y'$$
$$E'_z = \gamma E_z \qquad z = z'$$

because S' moving in $-x$ direction



$$x = \gamma R_x$$
$$y = R'_y$$
$$z = R_z$$

$\vec{R}$ is vector ~~from position~~ ~~of q to~~ from instantaneous position of q to observation point P

P 3

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \gamma q \vec{R} \Big/ \left(\gamma^2 R^2 \cos^2\theta + R^2 \sin^2\theta\right)^{3/2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q(1 - v^2/c^2)\hat{R}}{(1 - v^2/c^2 \sin^2\theta)^{3/2}} \frac{1}{R^2}$$

$\vec{E}$ is still parallel to $\vec{R}$ pointing radially away from instantaneous position of charge: Balancing of γ in $E_\perp$ and γ in $\vec{r}_\parallel$.

F4

This was just one special case $\vec{B} = 0$. The general law is

$$E_x' = E_x \qquad B_x' = B_x$$

$\vec{V}$ along $\hat{x}$ direction

$$E_y' = \gamma(E_y - vB_z) \qquad B_y' = \gamma(B_y + \tfrac{v}{c^2} E_z)$$

$$E_z' = \gamma(E_z + vB_y) \qquad B_z' = \gamma(B_z - \tfrac{v}{c^2} E_y)$$

reduce to our special capacitor case when $\vec{B} = 0$

Special cases $\vec{B} = 0$ in S

$$\vec{B}' = \gamma \frac{v}{c^2}\left(E_z \hat{y} - E_y \hat{z}\right) = \gamma \frac{v}{c^2}\left(E_z' \hat{y} - E_y' \hat{z}\right)$$

$$\vec{B}' = -\frac{1}{c^2}(\vec{V} \times \vec{E}')$$

$\hat{x}$	$\hat{y}$	$\hat{z}$
V_x	0	0
E_x'	E_y'	E_z'

$\vec{E} = 0$ in S

$$\vec{E}' = \vec{V} \times \vec{B}'$$

If $\vec{E} = 0$ or $\vec{B} = 0$ at some point in frame S

Then the fields in S' are simply related

F5

Evidently $\vec{E}$ and $\vec{B}$ do not transform simply

like 4 vectors $(ct, \vec{r})$ and $(E/c, \vec{p})$

because they mix. Generalize

$$(a^\mu)' = \Lambda^\mu{}_\nu a^\nu \qquad a^\mu = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}$$

"tensor" $(t^{\mu\nu})' = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma t^{\rho\sigma}$

display in 4×4 array $\begin{pmatrix} t^{00} & t^{01} & t^{02} & \\ t^{10} & t^{11} & & \\ & & & \\ & & & \end{pmatrix}$

The field tensor

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

combines $\vec{E}$ and $\vec{B}$ into a single quantity

just as $t, \vec{r}$ or $E, \vec{p}$ combined

Completely equivalently $\quad G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$

F6

Supplemental HW #1: Show $(F^{\mu\nu})' = (\Lambda^{\mu}{}_{\rho})(\Lambda^{\nu}{}_{\sigma})F^{\rho\sigma}$

gives the stated rules for $\vec{E}, \vec{B}$ transformation

How to write Basic Eqns of EM in terms of $F^{\mu\nu}$?

Start with continuity eqn

$$\vec{\nabla} \cdot \vec{J} = -\partial \rho / \partial t$$

$$J^{\mu} \equiv (c\rho, J_x, J_y, J_z)$$

$$\frac{\partial}{\partial x^{\mu}} J^{\mu} = 0$$

"current density 4 vector is divergenceless"

Supplemental HW #2 Maxwell Eqns are

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \mu_0 J^{\mu}$$

$$\frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0$$

Supplemental HW #3 $F^{\mu\nu} F_{\mu\nu} = ?$