Jackson Chapter 6

Maxwell Eqns before displacement current

$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ $\quad \vec{\nabla}\cdot\vec{D} = \rho \qquad \vec{D} = \epsilon_0\vec{E}$

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ $\quad \vec{\nabla}\times\vec{H} = \vec{J} \qquad \vec{B} = \mu_0\vec{H}$

$$\vec{\nabla}\times\vec{E} + \partial\vec{B}/\partial t = 0 \qquad \partial\rho/\partial t + \vec{\nabla}\cdot\vec{J} = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

continuity Forces

Faulty Eqn $\quad \vec{\nabla}\times\vec{H} = \vec{J}$ $\qquad (\vec{\nabla}\times\vec{H})_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}H_k$

$$\vec{\nabla}\cdot\vec{J} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = 0 \qquad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = \frac{\partial}{\partial x_i}\epsilon_{ijk}\frac{\partial}{\partial x_j}H_k$$

contradicts continuity.

Start with continuity

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = \vec{\nabla}\cdot\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right) = 0 \qquad \left(\frac{\partial\vec{D}}{\partial t} = \epsilon_0\,\partial\vec{E}/\partial t\right)$$

$$\vec{J} \to \vec{J} + \partial\vec{D}/\partial t$$

$$\boxed{\vec{\nabla}\times\vec{H} = \vec{J} + \partial\vec{D}/\partial t} \quad \text{Corrected Maxwell}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\,\partial\vec{E}/\partial t$$

6-1A

Often one then derives wave eqn for $\vec{E}, \vec{B}$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) + \underbrace{\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}}_{\frac{\partial}{\partial t}\mu_0\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)} = 0$$

$\vec{J} \to$ ($\emptyset$ in free space)

also $\vec{D} = \epsilon_0 \vec{E}$

$$(\vec{\nabla}\times\vec{E})_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}E_k$$

$$[\vec{\nabla}\times(\vec{\nabla}\times\vec{E})]_n = \epsilon_{nli}\frac{\partial}{\partial x_l}\epsilon_{ijk}\frac{\partial}{\partial x_j}E_k$$

$$\epsilon_{nli}\epsilon_{ijk} = \epsilon_{inl}\epsilon_{ijk}$$

$$= (\delta_{nj}\delta_{lk} - \delta_{nk}\delta_{lj})$$

$$[\vec{\nabla}\times(\vec{\nabla}\times\vec{E})]_n = \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_n}E_l - \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_l}E_n$$

$$= [\vec{\nabla}(\vec{\nabla}\cdot\vec{E})]_n - (\nabla^2 E)_n$$

$\emptyset$ in free space

$$-\nabla^2\vec{E} + \underbrace{\mu_0\epsilon_0}_{1/c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

Jackson focuses on Potentials $\vec{A}, \phi$

vector potential $\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$

Faraday $\vec{\nabla} \times \vec{E} + \partial \vec{B}/\partial t = 0$

$\rightarrow \quad \vec{\nabla} \times (\vec{E} + \partial \vec{A}/\partial t) = 0$

$\vec{E} + \partial \vec{A}/\partial t = -\nabla \phi$

$\nabla \times \nabla \phi = 0$

$[\nabla \times \nabla \phi]_i$

$= \epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \phi$

$$\boxed{\vec{E} = -\vec{\nabla} \phi - \partial \vec{A}/\partial t}$$

Four Maxwell's Eqns couple $\vec{B}$ and $\vec{E}$. Let's reduce to Eqns for $\vec{A}, \phi$

$\vec{\nabla} \cdot \vec{D} = \rho \qquad \vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$

$\vec{\nabla} \cdot (-\vec{\nabla} \phi - \partial \vec{A}/\partial t) = \rho/\epsilon_0$

$$\boxed{\nabla^2 \phi + \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} = -\rho/\epsilon_0}$$

$\vec{\nabla} \times \vec{H} = \vec{J} + \partial \vec{D}/\partial t \Rightarrow \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \frac{\partial \vec{D}}{\partial t}$

$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t} \right)$

$- \vec{\nabla}^2 A + \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = \mu_0 \vec{J} + \frac{1}{c^2} \vec{\nabla} \frac{\partial \phi}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2}$

$$\boxed{\nabla^2 A - 1/c^2 \frac{\partial^2 A}{\partial t^2} - \nabla \left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = -\mu_0 \vec{J}}$$

LT-1

$$\bar{B} = \bar{\nabla} \times \bar{A}$$
$$\vec{E} = -\partial A/\partial t - \nabla \phi$$

Last time we attempted to reconstruct Maxwell Eqns in terms of fields

$$\nabla^2 \phi - \partial/\partial t \, \vec{\nabla} \cdot \bar{A} = -\rho/\epsilon_0$$

$$\nabla^2 A - 1/c^2 \, \partial^2 A/\partial t^2 - \nabla(\nabla \cdot A + 1/c \, \partial\phi/\partial t) = -\mu_0 J$$

But this was ugly

Lorentz gauge $\quad \vec{\nabla} \cdot \vec{A} + 1/c^2 \, \partial\phi/\partial t = 0$

$$\nabla^2 \phi - 1/c^2 \, \partial^2\phi/\partial t^2 = -\rho/\epsilon_0$$

$$\nabla^2 A - 1/c^2 \, \partial^2 A/\partial t^2 = -\mu_0 J$$

$$\vec{A} \to \vec{A}' = \vec{A} + \nabla\Lambda$$

$$\phi \to \phi' = \phi - \partial\Lambda/\partial t$$

Another big simplification is $\vec{\nabla} \cdot \vec{A} = 0$

Coulomb gauge

Mathematically more complicated

5-1

Physics Interlude

$\boxed{\text{Schroedinger Eqn}}$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + q\phi\psi = -\hbar/i \frac{\partial\psi}{\partial t}$$

$\psi \to \psi' \equiv e^{i\Lambda}\psi$ Λ = indep of $\vec{r}, t$

ψ' also soln of Sch. Eqn.

Physics unchanged: $|\psi(\vec{r},t)|$ = prob to find particle @ $\vec{r}, t$ is the same

What if we wanted $\psi \to \psi' \equiv e^{i\Lambda(r,t)}\psi$

to give us a soln to Schroedinger eqn?

It would not work!

$$-\frac{\hbar^2}{2m}\nabla^2\psi' + q\phi\psi' \neq -\frac{\hbar}{i}\frac{\partial\psi'}{\partial t}$$

But if there were additional terms in Sch. Eqn

$$\frac{1}{2m}(\vec{p} - q\vec{A})^2\psi + q\phi\psi = -\frac{\hbar}{i}\frac{\partial\psi}{\partial t}$$

$\vec{p} = \frac{\hbar}{i}\vec{\nabla}$

$$\psi \to \psi' \equiv e^{i\frac{q}{\hbar}\Lambda(r t)}\psi$$

$$\vec{A} \to A' = \vec{A} + \vec{\nabla}\Lambda$$

$$V \to V' = V - \frac{\partial\Lambda}{\partial t}$$

Pretty Amazing

"Aesthetic" desire for qm to allow local

changes of phase $\Rightarrow$ Fields A, ϕ must exist!

Infer EM fields from desire for a particular

symmetry in QM.

other Examples of this in QM

Desire QM to be relativistically invariant ie

obey Lorentz symmetries $\rightarrow$ Dirac eqn

$\rightarrow$ electrons have spin!

still have $\vec{A}, \phi$ coupled in the two eqns

$$\nabla^2 \phi + \partial/\partial t \, \vec{\nabla} \cdot \vec{A} = -\rho/\epsilon_0$$

$$\nabla^2 A - 1/c^2 \, \partial^2 A/\partial t^2 - \nabla(\nabla \cdot A + 1/c^2 \, \partial \phi/\partial t) = -\mu_0 J$$

Gauge transformations

$$\boxed{\begin{aligned} \vec{A} \to \vec{A}' &= \vec{A} + \nabla \Lambda \\ \phi \to \phi' &= \phi - \frac{\partial \Lambda}{\partial t} \end{aligned}}$$

← Analog of arbitrary constant of integration in single variable calculus

$F = -\frac{dU}{dx}$

$U \to U' + C$

leave $\vec{B} = \vec{\nabla} \times \vec{A}$ $\quad \vec{E} = -\nabla \phi - \partial A/\partial t$ unchanged

We use this freedom to pick $\vec{A}, \phi$ to obey the Lorenz condition

$$\vec{\nabla} \cdot \vec{A} + 1/c^2 \, \partial \phi/\partial t = 0$$

As a consequence, get 2 uncoupled eqns $\phi, \vec{A}$

$$\boxed{\begin{aligned} \nabla^2 \phi - \tfrac{1}{c^2}\, \partial^2\phi/\partial t^2 &= -\rho/\epsilon_0 \\ \nabla^2 A - \tfrac{1}{c^2}\, \partial^2 A/\partial t^2 &= -\mu_0 \vec{J} \end{aligned}}$$

How do we arrange Lorenz condition

$$\vec{\nabla}\cdot\vec{A}' + \tfrac{1}{c^2}\, \partial\phi'/\partial t$$

$$= \vec{\nabla}\cdot\vec{A} + \tfrac{1}{c^2}\, \partial\phi/\partial t + \nabla^2\Lambda - \tfrac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}$$

so must find Λ obeying

$$\nabla^2\Lambda - \tfrac{1}{c^2}\, \partial^2\Lambda/\partial t^2 = -\left(\vec{\nabla}\cdot\vec{A} + \tfrac{1}{c^2}\, \partial\phi/\partial t\right) = f(x,t)$$

Once A', ϕ' found which satisfy Lorenz,

can further adjust them by any $\tilde{\Lambda}$ obeying

$$\nabla^2\tilde{\Lambda} - \tfrac{1}{c^2}\, \partial^2\tilde{\Lambda}/\partial t^2 = 0$$

Comments

(1) This eqn turns out to be solvable for any "reasonable" $f(\vec{r},t)$

(2) The gauge transformation Λ needed to get $\vec{A}, \phi$ to obey wave eqn itself obeys wave eqn

(3) Even after Λ is found which solves gauge eqn can do additional $\tilde{\Lambda}$ "restricted gauge transformations"

Mathematical interlude – the "Helmholtz Decomposition Theorem"

Any vector field $\vec{F}(x)$ can be decomposed into

$$\vec{F}(x) = \vec{F}_\ell(x) + \vec{F}_t(x)$$

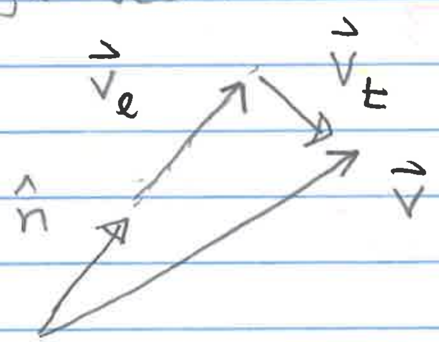
where $\vec{\nabla} \cdot \vec{F}_t = 0$ longitudinal

transverse

$\vec{\nabla} \times \vec{F}_\ell = 0$ Put another way we can write

$$\vec{F} = \underbrace{-\vec{\nabla} u}_{\nabla \times (\,) = 0} + \underbrace{\nabla \times \vec{W}}_{\nabla \cdot (\,) = 0}$$

Single vector



$$\vec{v} = \underbrace{(\vec{v} \cdot \hat{n})\hat{n}}_{\substack{\text{longitudinal} \\ \vec{v}_\ell}} + \underbrace{\vec{v} - (\vec{v} \cdot \hat{n})\hat{n}}_{\substack{\text{transverse} \\ \vec{v}_t}}$$

$\hat{n} \cdot \vec{v}_t = \phi$ obviously $\vec{v}_t$ is $\perp$ to $\hat{n}$

$\hat{n} \times \vec{v}_\ell = \phi$ $\vec{v}_\ell$ is $\parallel$ to $\hat{n}$

Note that if a field $\vec{F}$ obeys $\vec{\nabla} \cdot \vec{F} = 0$

and $\vec{\nabla} \times \vec{F} = 0$ it is not true that $\vec{F} = 0$

unlike our vector example $\hat{n} \cdot \vec{v} = 0$

$\Rightarrow |\vec{v}| = 0$ or $\cos\theta = 0$

$\theta = \pi/2$

$\hat{n} \times \vec{v} = 0$

$\Rightarrow |\vec{v}| = 0$ or $\sin\theta = 0$

so if both true $|\vec{v}| = 0$.

What is true is that $\vec{F} = -\nabla\phi$

where $\nabla^2 \phi = 0$

$\vec{F}$ is said to be "Laplacian"

However, if $\vec{\nabla} \cdot \vec{F} = 0$ and $\vec{\nabla} \times \vec{F} = 0$

and $\vec{F}$ vanishes at infinity then $\vec{F} = 0$.

Example

$$\vec{F} = (x-y)\hat{\imath} + (x+y)\hat{\jmath} + 4\hat{\imath} + 2\hat{\jmath}$$

transverse part F_t — longitudinal part F_ℓ — laplacian part

$$= (-y\hat{\imath} + x\hat{\jmath}) + (x\hat{\imath} + y\hat{\jmath}) + (4\hat{\imath} + 2\hat{\jmath})$$

$$\vec{\nabla}\times\vec{F} = 2\hat{k} \qquad 0 \qquad 0$$

$$\vec{\nabla}\cdot\vec{F} = 0 \qquad 2 \qquad 0$$

This Laplacian part is sort of trivial, a constant.

transverse part (zero divergence)

longitudinal part (zero curl)

laplacian part $-\vec{\nabla}(-4x - 2y)$

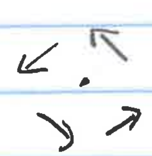
A less trivial laplacian part

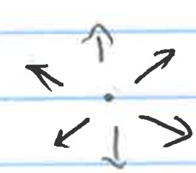
$$\phi = -\tfrac{1}{2}(x^2 - y^2) \qquad -\vec{\nabla}\phi = +x\hat{x} + y\hat{y}$$

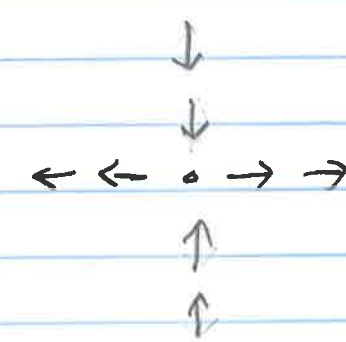
vanishing curl and gradient

$$\nabla^2\phi = 0$$

Pictures







these are what our example looks like

$\vec{B}$ due to wire

transverse part

$\nabla\cdot F_t = 0$

$\vec{E}$ due to pt charge

longitudinal part

$\nabla\times F_\ell = 0$

a more complicated laplacian part

Coulomb Gauge
Radiation Gauge
Transverse Gauge

$$\left.\begin{array}{l}\text{Coulomb Gauge}\\ \text{Radiation Gauge}\\ \text{Transverse Gauge}\end{array}\right\} \quad \vec{\nabla}\cdot\vec{A} = 0$$

Coulomb gauge is most convenient one for time independent problems

Then ϕ obeys Poisson eqn

$$\nabla^2\phi + \partial/\partial t\,(\vec{\nabla}\cdot\vec{A}) = -\rho/\epsilon_0$$

$$\longrightarrow \quad \nabla^2\phi = -\rho/\epsilon_0$$

$$\phi(x,t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x',t)\,d^3x'}{|x-x'|}$$

instantaneous Coulomb potential.

If working in Coulomb gauge strategy is then to solve this eqn for ϕ and then with ϕ known solve the eqn for $\vec{A}$

What about $\vec{A}$?

If we abandon Lorenz $\vec{A}$ remains coupled to ϕ

$$\nabla^2 A - 1/c^2\,\partial^2 A/\partial t^2 = -\mu_0\vec{J} + 1/c^2\,\vec{\nabla}\,\partial\phi/\partial t$$

6-8A ←

Write $\vec{J} = \vec{J}_\ell + \vec{J}_t$ with $\vec{\nabla}\times\vec{J}_\ell = 0$

Any vector field can be broken up this way

6-8B ←

Note that

$$\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta(\vec{x}-\vec{x}')$$

Green's function for Laplacian operator

$$\nabla^2 \Phi(x,t) = \nabla^2 \left\{ \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x',t)}{|x-x'|} d^3x' \right\}$$

$$= \frac{1}{4\pi\epsilon_0} \left\{ \int -4\pi\,\delta(x-x')\,\rho(x',t)\,d^3x' \right\}$$

$$= -\rho(x,t)/\epsilon_0$$

We will, shortly, work out Green's function

$G(x-x', t-t')$ for $\nabla^2 - 1/c^2\, \partial^2/\partial t^2$ operator.

A key feature will be causality.

Charge density at position x' at time t cannot affect Φ

at position x at same time t!

6-8.B

Several ways to prove. Last quarter

$$\vec{F}(\vec{x}) = \int d^3\vec{k}\ \vec{F}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$$

$$\uparrow$$

$$\hat{k}\left(\vec{F}(\vec{k})\cdot\hat{k}\right) + \left[\vec{F}(\vec{k}) - \hat{k}\left(\vec{F}(\vec{k})\cdot\hat{k}\right)\right]$$

$$\vec{F}(\vec{x}) = \vec{F}_\ell(\vec{x}) + \vec{F}_t(\vec{x})$$

$$\vec{F}_\ell(\vec{x}) \equiv \int \hat{k}\left(\vec{F}(\vec{k})\cdot\hat{k}\right) e^{i\vec{k}\cdot\vec{x}}\, d^3k$$

$\vec{\nabla}\times\vec{F}_\ell(x)$ involves $\vec{k}\times\hat{k} = 0$

$$\vec{F}_t(\vec{x}) \equiv \int \left[\vec{F}(\vec{k}) - \hat{k}\left(\vec{F}(\vec{k})\cdot\hat{k}\right)\right] e^{i\vec{k}\cdot\vec{x}}\, d^3k$$

$\vec{\nabla}\cdot\vec{F}_t(x)$ involves $\vec{k}\cdot\vec{F} - \underbrace{\underbrace{\vec{k}\cdot\hat{k}}_{|k|}(\vec{F}\cdot\hat{k})}_{\vec{F}\cdot\vec{k}}$

Another Look at Helmholtz

How is this decomposition actually accomplished

Notation 3D $x = \vec{x}$ $x' = \vec{x}'$

$$\vec{F}(x) = \int_V F(x')\,\delta(x-x')\,dx'$$

$$-\frac{1}{4\pi}\nabla^2\frac{1}{|x-x'|}$$

$$\vec{F}(x) = -\frac{1}{4\pi}\nabla^2\int_V \frac{F(x')}{|x-x'|}\,dx'$$

$$\nabla^2\vec{F} = \nabla(\nabla\cdot\vec{F}) - \nabla\times(\vec{\nabla}\times\vec{F})$$

$$\vec{F}(x) = -\frac{1}{4\pi}\nabla\left(\nabla\cdot\int_V \frac{F(x')}{|x-x'|}\,dx'\right)$$

$$+\frac{1}{4\pi}\nabla\times\left(\nabla\times\int_V \frac{F(x')}{|x-x'|}\,dx'\right)$$

longitudinal (curl vanishes) transverse (divergence vanishes)

$$\therefore\ \vec{F} = \underbrace{\vec{\nabla} u}_{\text{longitudinal}} + \underbrace{\vec{\nabla}\times\vec{w}}_{\text{transverse}}$$

with

$$u \equiv -\frac{1}{4\pi}\nabla\cdot\int_V \frac{\vec{F}(x')}{|x-x'|}\,dx'$$

$$\vec{w} \equiv \frac{1}{4\pi}\nabla\times\int \frac{F(x')}{|x-x'|}\,dx'$$

sometimes written as

$$= \frac{1}{4\pi}\int_V \frac{D(x')}{|x-x'|}\,dx'$$

$$= \frac{1}{4\pi}\int \frac{\vec{C}(x')}{|x-x'|}\,dx'$$

with $D = \vec{\nabla}\cdot\vec{F}$

$\vec{C} = \vec{\nabla}\times\vec{F}$

In the Coulomb gauge

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0 \vec{J} + \frac{1}{c^2}\nabla\frac{\partial\phi}{\partial t}$$

If we divide up $\vec{J} = \vec{J}_\ell + \vec{J}_t$

which Helmholtz tells us we can do

$$\vec{\nabla}^2 A - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0 \vec{J}_t$$

So even in Coulomb gauge $\vec{A}$ is okay with respect to causality

Why? Because

$$-\mu_0 \vec{J}_\ell = +\nabla\left[+\frac{\mu_0}{4\pi}\int\frac{\vec{\nabla}\cdot\vec{J}}{|x-x'|}\,dx'\right] \qquad \vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

$$= -\nabla\left[+\frac{\mu_0}{4\pi}\int\frac{\partial\rho/\partial t}{|x-x'|}\,dx'\right]$$

$$= -\mu_0\nabla\frac{\partial}{\partial t}\underbrace{\frac{1}{4\pi}\int\frac{\rho}{|x-x'|}\,dx'}_{\epsilon_0\phi}$$

$$= -\mu_0\epsilon_0\nabla\frac{\partial\phi}{\partial t} = -\frac{1}{c^2}\nabla\frac{\partial\phi}{\partial t}$$

So $\vec{J}_\ell$ piece cancels ϕ piece

What about ϕ. Does it violate causality? Yes, but can show the fields $\vec{E}, \vec{B}$ are causal.

Note: We saw (Aharonov-Bohm) that $\vec{A}$ is observable, but it is cons field) the potentials in the Coulomb gauge must satisfy the equations of motion:

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \tag{8.64}$$

$$\nabla^2 \boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J} + \frac{1}{c^2}\boldsymbol{\nabla}\frac{\partial \phi}{\partial t} \tag{8.65}$$

The potential ϕ is therefore the *well-known solution*

$$\phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{x}_0)}{|\boldsymbol{x} - \boldsymbol{x}_0|} d^3x_0 \tag{8.66}$$

that you probably originally saw in elementary introductory physics and solved extensively last semester using the Green's function for the Poisson equation:

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = -\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|} \tag{8.67}$$

that solves the "point source" differential equation:

$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0) \tag{8.68}$$

In this equation one uses the value of the charge density on all space as a function of time under the integral, and then adds a source term to the current density in the inhomogeneous wave equations for the vector potential derived from that density as well.

There are several very, very odd things about this solution. One is that the Coulomb potential is *instantaneous* - changes in the charge distribution *instantly* appear in its electric potential throughout all space. This appears to violate causality, and is definitely not what is physically observed. Is this a problem?

The answer is, no. If one works very long and tediously (as you will, for your homework) one can show that the current density can be decomposed into two pieces -- a *longitudinal* (non-rotational) one and a *transverse* (rotational) one:

$$\boldsymbol{J} = \boldsymbol{J}_\ell + \boldsymbol{J}_t \tag{8.69}$$

These terms are defined by:

$$\boldsymbol{\nabla} \times \boldsymbol{J}_\ell = 0 \tag{8.70}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{J}_t = 0 \tag{8.71}$$

Evaluating these pieces is fairly straightforward. Start with:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{J}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{J}) - \nabla^2 \boldsymbol{J} \tag{8.72}$$

This equation obviously splits into the two pieces – using the continuity equation to eliminate the divergence of $\boldsymbol{J}$ in favor of ρ, we get:

$$\nabla^2 \boldsymbol{J}_t = -\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{J}) \tag{8.73}$$

$$\nabla^2 \boldsymbol{J}_\ell = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{J}) = -\boldsymbol{\nabla}\frac{\partial \rho}{\partial t} \tag{8.74}$$

(which are both Poisson equations).

With a *bit* of work – some integration by parts to move the $\boldsymbol{\nabla}$'s out of the integrals which imposes the constraint that $\boldsymbol{J}$ and ρ have compact support so one can ignore the surface term – the decomposed currents are:

$$\boldsymbol{J}_t = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \int \frac{\boldsymbol{J} d^3 x_0}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|}) \tag{8.75}$$

$$\boldsymbol{J}_\ell = \boldsymbol{\nabla}\frac{\partial}{\partial t}\left(\int \frac{\rho}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|} d^3 x_0\right) = \epsilon_0 \boldsymbol{\nabla}\frac{\partial \phi}{\partial t} \tag{8.76}$$

Substituting and comparing we note:

$$\frac{1}{c^2}\boldsymbol{\nabla}\frac{\partial \phi}{\partial t} = \mu_0 \boldsymbol{J}_\ell \tag{8.77}$$

so that this term *cancels* and the equation of motion for $\boldsymbol{A}$ becomes:

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J}_t \tag{8.78}$$

only.

In the Coulomb gauge, then, only the *transverse current* gives rise to the vector potential, which behaves like a wave. Hence the other common name for the gauge, the *transverse* gauge. It is also sometimes called the "radiation gauge" as only transverse currents give rise to purely transverse radiation fields far from the sources, with the static potential present but not giving rise to radiation.

Given all the ugliness above, why use the Coulomb gauge at all? There are a couple of reasons. First of all the actual equations of motion that must be solved are simple enough once one decomposes the current. Second of all,